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ABSTRACTS

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Numerical simulation of convection around circular cylinder interacting with fluid jet

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The problem we consider has many important applications, and it is of fundamental interest in comparison with impingement at a cylinder due to uniform flow.

We study two-dimensional problem of interacting cylinder with incompressible jet. The numerical model is based on finite-difference approximation of the Navier–Stokes equations, written in variables "stream function" "intensity of vorticity," and heat transfer equation. For solution of these equations the finale difference scheme (upwind differences) with compensation of the first order of the truncation error is used. For Poisson equation, connecting stream function and intensity of vorticity, modified tridiagonal algorithm with alternating direction scheme is used. On the velocities and intensity of vorticity free boundary conditions are imposed.

This scheme was tested in the cases of essential convection, impingement cylinder by uniform flow, by limited flow. Good coincidences with the experimental data has been achieved.

Fluid flows in unbounded strip: patterns, stability, evolution

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Consider viscous fluid flow generated by the action of an external force F on plane torus T^2, i.e., suppose that the fluid flow satisfies periodicity conditions in the both spatial directions x_1, x_2 with the periods $2\pi/\alpha$ and 2π respectively. The simplest external force suggested by A. N. Kolmogorov is $F_* = (\gamma \sin(x_2), 0)^t$ and since early 60th it is well known that for the basic steady state, called Kolmogorov flow, the instability threshold corresponds to $\alpha = 0$ and Reynolds number R_0 that depends on force F. However, bifurcation analysis of the basic steady state and study of the dynamics generated by Navier–Stokes system were performed under the assumption that $\alpha \neq 0$. This is not surprising since $\alpha = 0$ corresponds to the flow in unbounded strip $O = R^1 \times S^1$, and this means that only periodicity condition in x_2 is left.

Therefore, our goal is to study bifurcation problem in unbounded domain and to provide a detailed study of time-independent fluid-flow profiles (u_1, u_2) uniformly in x_1 close to basic steady state $\mathbf{u}_* = (U, 0)^t$. For this end, we consider the time-independent Navier–Stokes system, rewrite it in the form of the evolution in x_1 problem and study spatial dynamics of the time independent problem. The space of all bounded solutions of the problem near the x_1-independent Kolmogorov solution $\mathbf{u}_*$ fits into a 6-dimensional center manifold of spatial profiles. Even though the initial value problem for the time-independent Navier-Stokes system is elliptic and hence ill-posed, translation by x_1 induces an autonomous flow on this manifold. Our study of small bounded solutions of this reduced *spatial-dynamics flow*, where x_1-translation plays the role of a "time" action, will account for new homoclinic pulse-type and heteroclinic multipulse solutions to the Kolmogorov problem, near Kolmogorov's instability threshold. Spatial approach gives no hint about the stability of such patterns, but suggests the set of bounded uniformly continuous functions as a pertinent phase space. Hence, the first step of the stability analysis is to prove the solvability of the Cauchy problem with uniformly bounded initial data. A priori estimate of the non-stationary solutions is given in in the $L^\infty(O)$ norm. For the time independent of the force $F(x)$, this estimate is polynomial in time. From the estimate follows the global in time existence of solutions of the Cauchy problem. This statement enables to investigate the stability of the time independent kink and front solutions of the Navier–Stokes system.

Low-dimensional control of the 2D Navier–Stokes and Euler equations

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We consider the Navier–Stokes and Euler equations on the 2-dimensional Riemannian surface M homeomorphic to the sphere, torus or disc. In the last case we assume that ∂M is a piecwise smooth curve and impose the Lions boundary condition. The equations written in terms of the vorticity w and the stream functions ψ read:

$$\frac{\partial w}{\partial t} + \{\psi, w\} - \nu\Delta w = f(t,x), \quad \Delta\psi = w, \tag{1}$$

$$0 \le t \le T,\ x \in M, \quad \psi\big|_{\partial M} = w\big|_{\partial M} = 0,$$

where $\{\cdot,\cdot\}$ is the Poisson bracket, Δ the Laplace–Beltrami operator, ν a nonnegative real number, and the right-hand side f is the vorticity of the external force. We assume that the right-hand side has the form:

$$f(t,x) = f_0(t,x) + \sum_{i=1}^{k} v_i(t) e_i(x),$$

where $f_0(x)$ is the vorticity of a fixed external force, $e_1, \dots, e_k$ are eigenfunctions of Δ (i.e., some harmonics), and $v_1(\cdot), \dots, v_k(\cdot)$ are control functions in our disposition. We assume that $v_i(\cdot)$ belong to the *space of admissible controls* $V \subset L_\infty[0,T]$ and that V is an everywhere dense vector subspace of $L_1[0,T]$.

Given $\varphi_0 \in H_2(M)$, we say that $\varphi_T \in H_2(M)$ is reachable from φ_0 if there exist admissible control functions $v_1(\cdot), \dots, v_k(\cdot)$ such that the solution of system (1) with the initial condition $w(0,\cdot) = \varphi_0$ satisfies the equation $w(T,\cdot) = \varphi_T$. Let $\mathcal{R}_T(\varphi_0) \subset H_2(M)$ be the set of all reachable functions. We say that the system is L_2-*approximately controllable* if

$\mathcal{R}_T(\varphi_0)$ is everywhere dense in $L_2(M)$ for any $\varphi_0 \in H_2(M)$. The system is *controllable in finite-dimensional projections* if the L_2-orthogonal projection of $\mathcal{R}_T(\varphi_0)$ on any finite dimensional subspace of $H_2(M)$ is surjective. We also deal with *solid* controllability in finite dimensional projections that is a robust version of the usual one: we require the surjectivity property to survive small distortions of the data (see [1] for the precise definition).

It was shown in [1] for the case of a flat torus and in [2] for a flat rectangular that an appropriate selection of harmonics $e_1, \ldots, e_k$ guarantees all mentioned types of controllability (with $k = 4$ for the torus and $k = 8$ for the rectangular). The proves of these results heavily used the structure of nonlinear term $\{\psi, w\}$ and were actually based on the explicit coordinate expression of this term in the basis formed by the harmonics; therefore these proves could not be directly extended to other Riemannian surfaces.

In my talk, I will discuss the modified essentially more flexible approach to the problem which allows, in particular, to prove the L_2-approximate controllability and solid controllability in finite dimensional projections in the case of generic Riemannian structure for all 3 types of surfaces under consideration, with $k = 3$ for the sphere, $k = 4$ for the torus and $k = 8$ for the disc. Moreover, in the case of the disc the desired controllability properties are valid for generic *flat* Riemannian structure as well (i.e. for generic bounded simply connected domain in $\mathbb{R}^2$ with a good boundary). This is a joint work with A. A. Sarychev and S. Rodrigues.

If time permits, I will also discuss open questions and possible directions for further development of the topic.

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On a higher analog of hydrodynamic current helicity integral

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The Euler equation in ideal hydrodynamics implies the equation

$$\frac{d\chi^m}{dt} = 0, \quad \chi^m = \int (V, \operatorname{rot} V) dx,$$

where χ^m is called helicity. In a medium with non-vanishing small diffusion coefficient η the helicity satisfies the equation

$$\frac{d\chi^m}{dt} = 2\eta\chi^c, \quad \chi^c = \int (\operatorname{rot} V, \operatorname{rot} \operatorname{rot} V) dx,$$

where the integral in the right side of the equation is called the current helicity integral. This equation can be proved analytically and admits a topological interpretation.

We shall discuss a higher analog (i.e., an analog related to higher algebraic invariants of links and knots) of the currant helicity integrals. This integral is defined by means of an integral formula for the generalized Sato—Levine invariant, which was introduced by L. Traldi (1988), M. Polyak and O. Ya. Viro (1995), P. Kirk and C. Livingston (1997), D. Repovs and the author (1998), Nakanishi and Ohyama (2002) independently.

Theoretical analysis of control problems for stationary magnetohydrodynamic models of a viscous heat-conducting fluid

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Control problems for flows of an electrically and thermally conducting viscous fluid play an important part in a number of magnetohydrodynamic (MHD) applications. Among them are the creation of cooling systems using liquid metals for nuclear reactors in nuclear industry, the development of technologies for the noncontacting electromagnetic stir of molten metals in foundry industry, the creation of installations for industrial crystal growing in microelectronic industry, and the development of new submarine engines. One of the purposes of the simulation based on MHD models of a viscous heat-conducting fluid is to study the influence of a magnetic field on convection. In some cases (e.g., in the cooling of nuclear reactors), a magnetic field is used to amplify convection. On the contrary, in crystal growth installations, it is used to suppress convection because the latter deteriorates the quality of crystals. The exploration of the possibility of intensifying or suppressing convection by optimization methods leads to control problems for MHD models that seek the most efficient mechanisms for controlling thermal and hydrodynamic processes in a viscous heat-conducting fluid exposed to a magnetic field. Some of these problems are stated and analyzed in this paper by using the technique of [1–4].

This technique makes it possible to prove the solvability of the original nonhomogeneous boundary problems for the MHD models under consideration and control problems for a wide class of weakly lower semicontinous cost functionals, to derive optimality systems, and to prove the local uniqueness theorems of solutions to particular control problems. Detailed discussion of some results for MHD models of viscous heat-conducting fluid considered under mixed boundary conditions for the velocity can be found in [5].

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Free-surface problems in 3d fluids

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In prior work, I have shown that the various potential flows involving free surfaces in 2D fluids are well-posed in Sobolev spaces for short times. The particular problems treated are the vortex sheet with surface tension, Hele-Shaw flow, and (in joint work with N. Masmoudi) the irrotational water wave. The method used is (following the excellent numerical work of Hou, Lowengrub, and Shelley) to first reformulate the equations of motion using suitable, geometric variables and a renormalized arclength parameterization. Then, energy estimates are performed. In these problems, since the fluids are two-dimensional, the free surface is a curve.

In the case of three-dimensional fluids, the free surface is now a two-dimensional surface rather than a curve. There is thus no direct analogue of arclength available. We make a different, but still quite favorable, choice of coordinates and dependent variables. The equations of motion, when formulated in this manner, are amenable to energy estimates. After establishing the energy estimates, existence follows in a standard way. It should be mentioned that there are additional differences between the cases of 2D and 3D, especially that the singular integrals must be treated much more carefully in the higher-dimensional case.

In this talk, I will discuss the above issues for the vortex sheet with surface tension (this is joint work with N. Masmoudi and J. Shatah).

Asymptotic properties of solutions of the Navier–Stokes equations in 2D exterior domains

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In this talk, I will explain how techniques herited from dynamical systems theory can be applied to derive asymptotic properties for Navier–Stokes flows in 2D exterior domains. In both the stationary and time-periodic case, interpreting the coordinate parallel to the limiting (non-zero) velocity at infinity as a "time coordinate," allows one to write the 2D Navier–Stokes equations in a form which is reminiscent of a nonlinear parabolic equation in 1D, with the somewhat surprising result that both type of flows cannot be distinguished from one another far in the downstream direction.

Linear superposition of nonlinear waves

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Nonlinear waves are described by nonlinear differential equations. Their solutions are determined by initial data, which are functions of the spatial variables. When the equation is linear, if the initial function equals the sum of two or several functions, the solution equals the sum of corresponding solutions. For nonlinear equations this linear superposition principle is not valid. Nevertheless, there are important and physically relevant systems and classes of initial data for which the solution equals the sum of corresponding solutions with a small error. The examples include: Fermi–Pasta–Ulam system, nonlinear wave equation, nonlinear Schrodinger equation, Navier–Stokes and Euler systems in a rotating frame, Boussinesq system with a strong rotation or stratification. The equations have the form

$$\partial_\tau \mathbf{U} = -\nu \mathbf{A}\left(-i\nabla\right)\mathbf{U} - \frac{i}{\varrho}\mathbf{L}\left(-i\nabla\right)\mathbf{U} + \mathbf{F}\left(\mathbf{U}\right),$$
$$\mathbf{U}\left(\mathbf{r},\tau\right)|_{\tau=0} = \mathbf{h}\left(\mathbf{r}\right),\ \mathbf{r}\in\mathbb{R}^d,\ 0\le\tau\le\tau_*,$$

where $\nu \geq 0$, $\mathbf{A}$ is a positive differential or pseudodifferential operator which controls dissipation, $\mathbf{L}\left(-\mathrm{i}\nabla\right)$ is a self-adjoint differential or pseudodifferential operator which controls linear wave dynamics, $\mathbf{F}\left(\mathbf{U}\right)$ is the nonlinearity, $\varrho \ll 1$. We denote by $\mathbf{S}_\tau$ the solution operator which corresponds to this equation, $\mathbf{S}_\tau \mathbf{h} = \mathbf{U}\left(\tau\right)$. Under certain conditions

$$\mathbf{S}_\tau\left(\mathbf{h}_1 + ... + \mathbf{h}_N\right) = \mathbf{S}_\tau\left(\mathbf{h}_1\right) + ... + \mathbf{S}_\tau\left(\mathbf{h}_N\right) + \mathbf{D},\ \sup_{0\le\tau\le\tau_*}\left\|\mathbf{D}\right\|_{L_\infty} \le C\varrho^{1/2}.$$

The described approximate linear superposition of nonlinear waves is explained by a destructive interference between waves from different wavepackets in the process of their time evolution, this interference drastically reduces nonlinear interaction between the wavepackets. Note that nonlinear effects which determine evolution of every $\mathbf{S}_\tau\left(\mathbf{h}_j\right)$ are not small.

Boundary layer problems in 2d and 3d

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In the mathematical approach of fluid dynamic many other basic issues than the famous "millenium problem" remain unsolved (and even in $2d$). I have the feeling that the solutions of these problems would not be easier than the "millenium" one and that breakthrough in this questions may be bring the kind of tools that would be useful in other situations. Among these is the problem of the boundary layer for Navier–Stokes equation with no slip boundary condition and viscosity going to zero. Even in two space variables this is a very difficult problem with only very partial answer. It is related both to the understanding of turbulence and to many applications. Consider in a domain Ω the solution of the Navier–Stokes equation with viscosity $\nu > 0$

$$\partial_t u_\nu + \nabla(u_\nu \otimes u_\nu) + \nabla p_\nu = 0\,, \nabla \cdot u_\nu = 0$$

and with smooth divergence free, finite energy initial data. Assume that Ω has no boundary (either the whole space or the periodic box). Assume furthermore that for $0 < t < T$ the Euler equation (which corresponds to $\nu = 0$) with same initial data do have a smooth solution in $\Omega\times]0,T[$; then u_ν converges to this solution in $C(0,T;L^2(\Omega))$.

This is an old well known result but may be the most convenient proof is the use of the notion of dissipative solutions of R. Di Perna and P. L. Lions (cf. [7]). It should also be noticed that this notion of dissipative solution is a basic tool in the proof obtained by L. Saint-Raymond [8] for the convergence (under convenient scalings) of the renormalised solution of the Boltzmann equation to the solution of the Euler equation.

Since in 2 spaces variables the existence and uniqueness of a smooth solution is known this shows that u_ν converges to such solution. On the other hand in 3 spaces variable and this is probably the most basic open problem of the theory there is no (for large T) information concerning the existence of a (strong or weak) solution of the Euler equation. The only a priori estimate is the energy estimate:

$$\int_\Omega |u_\nu(x,t)|^2 dx + 2\nu \int_0^t \int_\Omega |\nabla_x \wedge u(x,t)|^2 dx \le \int_\Omega |u(x,0)|^2 dx \qquad (1)$$

From (1) one deduces a weak convergence of u_ν to a function $\overline{u} \in L^\infty(0,T; L^2(\Omega))$ but as it is well known this is not enough to prove that $\overline{u}$ is a solution of the Euler equation the reason being that the estimate (1) is not enough to imply the relation:

$$\lim_{\nu\to 0}(u_\nu \otimes u_\nu) = \lim_{\nu\to 0}(u_\nu) \otimes \lim_{\nu\to 0}(u_\nu)$$

Now one has

$$\lim_{\nu\to 0}(u_\nu \otimes u_\nu) - \lim_{\nu\to 0}(u_\nu) \otimes \lim_{\nu\to 0}(u_\nu) = \lim_{\nu\to 0}(u_\nu - \lim_{\nu\to 0} u_\nu) \otimes \lim_{\nu\to 0}(u_\nu - \lim_{\nu\to 0} u_\nu) \tag{2}$$

and the right hand side of (2) can be well described in term of a Wigner (cf. Gérard [5]) defect measure which would be the limit of the Fourier transform of the scaled correlation:

$$R(x,t,k) = \lim_{\nu\to 0}\int e^{iky} u_\nu\left(x + \frac{\sqrt{\nu}}{2}y, t\right) \otimes u_\nu\left(x - \frac{\sqrt{\nu}}{2}y, t\right) dy$$

At the present state of our knowledge one cannot be sure that $R(x,t,k)$ is in this situation (no boundary smooth initial data) identically zero but one should observe that this object is the natural deterministic counterpart of the spectra of a statistical turbulence:

$$R_{stat}(x,t,k) = \int e^{iky} \left\langle u_\nu\left(x + \frac{\sqrt{\nu}}{2}y, t\right) \otimes u_\nu\left(x - \frac{\sqrt{\nu}}{2}y, t\right)\right\rangle dy$$

for which a series of properties (homogeneity isotropy and inertial range) are part of the folklore of the theory.

For the Navier Stokes equation in a domain Ω with boundary $\partial\Omega$ with a "no slip boundary condition"

$$u_{\nu\,\partial\Omega} = 0 \tag{3}$$

the issue of the convergence of u_ν to the solution of the Euler equation (even if such solution exist and is smooth) is a completely open problem. Everyday experiments and numerical computations indicate that in principle this convergence does not hold everywhere in space time. For the solution of the Euler equation $\nu = 0$ the tangential component of the velocity is no more zero and the difference with (3) generates a boundary layer. Since the problem is non linear such boundary layer may propagate inside the fluid. It is in full agreement with the notions of detachment point and Von Karman vortex street found in the literature.

Criteria for convergence to the Euler equation (always valid in $2d$ and with the hypothesis of the existence of a smooth solution in $3d$) is (cf. Kato [6]) equivalent to:

$$\lim_{\nu\to 0} \nu \int_0^T \int_{x\in\Omega, d(x,\partial\Omega)} |\nabla \wedge u_\nu|^2 dxdt = 0\,. \tag{4}$$

which means that the production of the vorticity in a layer of size ν is $o(\nu^{-1})$. With the notion of dissipative solution one produces a weaker statement saying that (4) is equivalent to $\nu\nabla \wedge u_{\nu\partial\Omega}$ goes to zero in $\mathcal{D}(\partial\Omega\times]0,T[)$ One can also observe that the equation (4) gives for a boundary layer of the size $O(\nu)$ instead of $O(\sqrt{\nu})$ which corresponds to standard parabolic equations (Navier–Stokes is not parabolic). Furthermore, (4) is in agreement with the existence of an approximate solution given by the Prandlt equations. The hypothesis concerning the Prandlt equations stronger than (4), when they have smooth solutions this implies that the boundary layer remains confined at a distance of the order of ν from the boundary and that the convergence to the solution of the Euler equation holds. On the other the Prandlt system is very unstable ("more" than Kelvin Helmholtz or Rayleigh Taylor) and the proof of existence of solutions and convergence to solutions of the Euler equation has been obtained only with the very unstable hypothesis of analyticity (cf. Asano, Caflisch Sammartino and Canone [1–3]).

It is therefore natural to expect that even in $2d$ with the no slip boundary condition at the level of the Navier–Stokes equation:

$$\lim_{\nu\to 0}(u_\nu \otimes u_\nu) - \lim_{\nu\to 0}(u_\nu) \otimes \lim_{\nu\to 0}(u_\nu) \neq 0 \ \text{ or}$$
$$R(x,t,k) = \lim_{\nu\to 0} \int e^{iky} u_\nu \left(x + \frac{\sqrt{\nu}}{2}y, t\right) \otimes u_\nu \left(x - \frac{\sqrt{\nu}}{2}y, t\right) dy \neq 0$$

Since the production of this defect measure is a self sustained effect involving only high frequencies a natural conjecture would be to assume that such object are Galilean invariants. With this conjecture and simple geometrical considerations one would obtain for the limit equation

$$\partial_t u + \nabla(u \otimes u) - \nabla(\nu_{turb}(\nabla u + \nabla u^\perp)) + \nabla p = 0$$

which corresponds to the Smagorinskii model for turbulence.

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Singular sets. Decay of Fourier coefficients

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The singular set of a weak solution u for the Navier–Stokes equation consists of space-time points where u is locally unbounded. The famous Scheffer and Caffarelli–Kohn–Nirenberg theorem states that for suitable weak solutions the set of singular points can not be "too massive." Precisely this means that the (parabolic) Hausdorff 1D measure of this set must be zero.

One could try to derive from this that singular sets are generically disjoint. However, we have the following theorem.

Theorem 1. *Let u and v be two weak solutions of the same Cauchy problem for Navier–Stokes equation on the torus $\mathbb{T}^d$. Assume that their singular sets are disjoint. Then both sets are in fact empty and $u = v$ is the unique strong solution.*

The following uniform bounds for the Fourier coefficients is of independent interest.

Theorem 2. *Let u be a weak solution for the Navier–Stokes equation on the torus $\mathbb{T}^d$. Assume that $u(0) \in H^1(L^2)$. Then there exist C_1, C_2, C_3, such that the Fourier coefficients satisfy the following estimates*:

$$\sup_{t\in[0,\infty)} |\hat{u}_k(t)| < \frac{C_1}{|k|}\,, \quad \int_0^\infty |\hat{u}_k(t)|^2 dt < \frac{C_2}{|k|^4}\,, \quad \int_0^\infty |\hat{u}_k(t)| dt < \frac{C_3}{|k|^2}\,.$$

On the solutions of the Boltzmann equation with one-dimensional symmetry

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For the Boltzmann equation, the setting of a narrow shock tube implies that solutions depend on a single spatial variable, but may have a genuine three-dimensional dependence on the microscopic velocity. For such solutions we discuss the criteria under which strong solutions, with uniformly bounded macroscopic density, exist for all positive times. It was previously known that solutions with such properties were unique in the class of dissipative weak solutions in the sense of P.-L. Lions; however, their existence was established either locally in time, or near a global equilibrium given by a spatially independent Maxwellian distribution. The principal result of our paper is that for bounded Boltzmann collision kernels, with a cutoff for small relative speeds, the propagation of L^∞ bounds on the macroscopic averages is established globally in time. The crucial tools of our analysis are apriori bounds using relative entropy and a certain quadratic functional introduced originally by L. Tartar and J.-M. Bony and adapted to the Boltzmann equation by C. Cercignani. Using these bounds and an averaging effect of the collision operator we derive novel Gronwall-type inequalities which allow us to obtain global existence by a contraction argument, and in some cases yield quantitative bounds on the time growth of the L^∞ norms.

Invariants of Hamiltonian and bi-Hamiltonian systems of hydrodynamic type

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Differential-geometric invariants of Hamiltonian systems of partial differential equations of hydrodynamic type

$$u_t^i = \sum_{j=1}^{n} A_j^i(u^1, \cdots, u^n) u_x^j$$

are introduced. Algebraic identities for the corresponding Nijenhuis and Haantjes (1,2)-tensors are found. The necessary conditions for existence of a non-degenerate Hamiltonian structure for a given hydrodynamic type system are obtained in terms of the polynomial $P_H(v, \lambda) = \det(H_v - \lambda I)$, the special differential 2-forms $\Omega_{pq}(v_1, v_2)$ and k-forms $\Omega_{p_1 \cdots p_k}(v_1, \ldots, v_k)$. Applications to the perturbations of the Benney system are considered.

The necessary conditions are derived for existence of a bi-Hamiltonian structure for a given hydrodynamic type system. A theorem is proved on the canonical forms of the generic bi-Hamiltonian systems.

The boundary layer on a needle

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About hundred years ago, L. Prandtl and H. Blasius had created a theory of the boundary layer on a semi-infinite plate for the stationary flow of a viscous incompressible fluid. In Section 6 of Chapter 6 [1] that theory is treated from the Power Geometry view-point. It gives the pure mathematical justification of the boundary layer theory, which does not use any mechanical or physical arguments. The similar theory of a boundary layer on a semi-infinite needle is not created up to now. The theory of boundary layer on a long thin cylinder [2] has no limit if the radius of the cylinder tends to zero. In [3] we considered the flow of the viscous incompressible fluid around the needle by methods of Power Geometry and showed that this problem has no solution.

So we decided to consider the stationary spatial axially symmetric flow of the viscous compressible heat conducting gas around a semi-infinite needle. The flow is described by a system of three partial differential equations for stream function, the density and the enthalpy (analog of temperature) with two independent variables: along the symmetry axis and distance from the axis. The boundary conditions are given at infinity (an uniform filling flow) and at the needle (the adhesion condition). The truncated system of the equations describing flow in the boundary layer is selected by methods of Power Geometry [1]. After introduction of self-similar coordinates, the truncated system is reduced to a system of two ordinary differential equations. The system has the invariant manifold where the system is reduced to one ordinary differential equation of the second order. The asymptotical analysis of its solutions by methods of Power Geometry [4] shows, that the equation has solutions satisfying boundary conditions at infinity and at the needle. At the needle they have a logarithmic singularity. When we tends to infinity along the needle, then the density tends to zero and enthalpy tends to infinity. These solutions were found theoretically only. Except them, we have shown by numerical computation the existence of additional solutions of the ordinary differential equation. Near the needle they have the same singularities. For fixed values of parameters of the initial problem, the found solutions form a two-parameter family and a one-parameter family which is the boundary of the two-parameter family [5].

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Variational inequalities for evolution MHD systems

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The magnetohydrodynamic (MHD) equations in the bounded simply connected domain $\Omega \subset \mathbb{R}^d$ with the connected boundary Γ are considered.

$$\partial u/\partial t - \nu \Delta u + \operatorname{rot} u \times u = -\nabla h + S \cdot \operatorname{rot} B \times B,\ x \in \Omega,\ t > 0, \tag{1}$$

$$\partial B/\partial t + \operatorname{rot} E = 0,\ j = \operatorname{rot} B = 1/\nu_m (E + u \times B), \tag{2}$$

$$\operatorname{div} u = 0,\ \ \operatorname{div} B = 0. \tag{3}$$

Here u, B, E and j are vector fields of velocity, magnetic induction, electric intensity and current density respectively; h is total flow pressure, $\nu = 1/Re$. $\nu_m = 1/R_m$, $S = M^2/Re\, R_m$, where Re,Re_m and M are the Reynolds number, Reynolds magnetic number and Hartmann number.

To the equations (1)–(3) we add initial conditions

$$u|_{t=0} = u^0(x),\ B|_{t=0} = B^0(x),\ x \in \Omega. \tag{4}$$

and the conditions on the boundary Γ

$$u = 0,\ B \cdot n = 0 \qquad (x, t) \in \Gamma \times (0, T), \tag{5}$$

where n is the unit outward normal to the boundary. The boundary conditions (5) should be supplemented by the conditions describing interaction of the electric and magnetic fields on the boundary. Usually either tangential components of the electric field or rotor of the magnetic field are prescribed. Here we consider "energetic" boundary conditions in a form of the variational principle. Note that the expression $\int_\Gamma (n \times E) B d\Gamma$ is proportional to the work per unit time of the electric field on the surface currents. Furthermore, from the first equation in (2) it follows that

$$\int_\Gamma (n \times E) \nabla \zeta d\Gamma = 0$$

and therefore only a rotor part $R(B)$ of the boundary magnetic field affects this work. For the tangent vector field $B|_\Gamma$ the Weyl decomposition $B =$

$n \times \nabla r + \nabla q$ in a sum of the orthogonal $L^2(\Gamma)$ fields is valid. Hence $R(B) = n \times \nabla r$.

The problem is to find a solution to the system (1)–(3) under conditions (4), (5) and additional subdifferential inclusion

$$n \times E \in R(\partial\Psi(R(B), x)), \quad x \in \Gamma. \tag{6}$$

Here $\Psi : \mathbb{R}^d \times \Gamma \to \mathbb{R}$ is a convex lower semicontinuous in the first argument function and $\partial\Psi$ is its subdifferential in the first argument.

The theory of solvability of an abstract evolution inequality in a Hilbert space for the operators with the quadratic nonlinearity is created. Obtained results is used for the study of MHD flows. For the 3-dimensional flows the global in time existence of the weak solutions to the variational inequalities is proved. For the two-dimensional flows existence and uniqueness of the strong solutions are proved.

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On the relation between the trajectory attractors of the Camassa–Holm system and the 3D Navier–Stokes system

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We study the connection between the long-time dynamics of the Navier–Stokes-α model (also known as the Lagrangian averaged Navier–Stokes-α or Camassa–Holm system) and the 3D Navier–Stokes system with periodic boundary conditions. It was demonstrated analytically and computationally in many works that the Navier–Stokes-α model is a powerful tool in the study of the sub-grid effects in the motion of large eddy currents. Recently, it was proved that the Cauchy problem for the equations of Navier–Stokes-α model has a unique solution in the corresponding function space.

We consider bounded (in the energy norm) families of solutions of the Navier–Stokes-α model for $0 < \alpha \leq 1$. For $\alpha = 0$, we formally have the classical 3D Navier–Stokes system for which the uniqueness theorem (on the entire time semi-axis) of the existing weak solution of the Cauchy problem is not proved yet. However, for the 3D Navier–Stokes system, we can construct the trajectory attractor. The trajectory attractor is the set of all weak solutions of the system that are bounded in the energy norm, satisfy the main energy inequality and have bounded prolongations on the negative semi-axis as weak solutions of the original system keeping the energy inequality.

We prove that time shifts $\{T(h), h \geq 0\}$ (here $T(h)w(t) = w(t+h)$) of bounded sets of solutions $B_\alpha = \{w(t), t \geq 0\}$ of the Navier–Stokes-α model approach the trajectory attractor $\mathfrak{A}$ of the 3D Navier–Stokes system in the corresponding topology as h tend to $+\infty$ and $\alpha \to 0+$. In particular, we show that the trajectory attractor $\mathfrak{A}_\alpha$ of the Navier–Stokes-α model converges to the trajectory attractor $\mathfrak{A}$ of the 3D Navier–Stokes system as $\alpha \to 0+$.

The results of my talk is based on our joint work with M. I. Vishik and E. S. Titi.

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Counter-examples to the concentration-cancellation property

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The framework of this talk is given by the two dimensional incompressible Euler equation

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_x)\mathbf{u} + \nabla_x \mathbf{p} = 0 \,, \qquad \mathrm{div}_x \, \mathbf{u} = 0 \tag{1}$$

or by the two dimensional Burger equation

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_x)\mathbf{u} + \mathbf{f} = 0 \tag{2}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^2$ and $\mathbf{f}(t,x)$ is a source term.

Fix some time $T > 0$ and some open domain $\Omega \subset \mathbb{R}^2$. Consider a family of functions $\{\mathbf{u}^\varepsilon\}_\varepsilon$ satisfying:

(i) $\mathbf{u}^\varepsilon(t,x) \in C^1([0,T] \times \Omega; \mathbb{R}^2)$ for all $\varepsilon \in]0,1]$,

(ii) $\mathbf{u}^\varepsilon(t,x)$ is a solution of (1) or (2) for all $\varepsilon \in]0,1]$,

(iii) There exists a constant C leading to the uniform control

$$\| \, \mathbf{u}^\varepsilon \, \|_{L^\infty([0,T]\times\Omega)} \leq C \,, \qquad \forall \varepsilon \in]0,1] \,.$$

The asymptotic behavior of such a solution sequence $\{\mathbf{u}^\varepsilon\}_\varepsilon$ can reveal (when ε goes to zero) various and complex phenomena (involving for instance concentrations and oscillations). A way to understand the phenomena which can happen in the limiting process is to perform a *nonlinear geometric optics under constraint*.

Basically, this means to study large amplitude high-frequency oscillating waves which:

(a) have (when ε goes to zero) the form $\mathbf{u}^\varepsilon(t,x) \sim \mathbf{U}^\varepsilon\big(t,x,\Phi^\varepsilon(t,x)/\varepsilon\big)$ where $\mathbf{U}^\varepsilon(t,x,\theta) \in C^1([0,T] \times \Omega \times \mathbb{T}; \mathbb{R}^2)$ (with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$) is some profile and $\Phi^\varepsilon(t,x) \in C^1([0,T] \times \Omega; \mathbb{R})$ is some phase to identify,

(b) are subjected (for all $\varepsilon \in]0,1]$) to the constraint $\mathbf{u}^\varepsilon(t,x) \in \mathcal{V}$ where $\mathcal{V}$ is some functional set to specify.

The aim of this conference is to give a rapid presentation of this approach and to illustrate it by applications.

Axisymmetric incompressible flows with bounded vorticity

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We are concerned with the Cauchy problem for the three-dimensional incompressible Euler equations:

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = 0, \\ \operatorname{div} v = 0, \end{cases} \tag{E}$$

in an open set Ω which is either the whole space $\mathbf{R}^3$ or a smooth bounded connected domain (in which case system (E) is supplemented with the slip boundary condition $v \cdot n = 0$ on $\partial\Omega$).

In dimension two, global existence and uniqueness for smooth data has been first stated in 1933 by W. Wolibner [4], then by V. Yudovich in 1963 for rough data with bounded vorticity [5].

The proof relies on the following two facts. First, the vorticity ω (which, in dimension two, reduces to a scalar function) is transported by the flow hence has a L^∞ norm independent of the time. Second, the corresponding velocity field v is quasi-lipschitz (i.e lipschitz up to a log), hence has a flow. Uniqueness stems from a stability estimate in energy norm which relies on a generalized Gronwall lemma.

In dimension three, the problem of global solvability is much more difficult as the vorticity (which may be identified with a solenoidal vector-field) is transported by the flow as a vector-field:

$$\partial_t \omega + v \cdot \nabla \omega = \omega \cdot \nabla v. \tag{1}$$

As a matter of fact, whether three-dimensional Euler equations have global solutions for general smooth data is open. There are however at least three listed cases where geometrical structure of the data is known to cause global existence:

- data with translational invariance (which is similar to the two-dimensional case),
- helicoidal data (that is the components of the velocity field in cylindrical coordinates are constant on a prescribed family of helicoids, see [1]),
- axially symmetric data *without swirl.*

In this talk, we shall mainly focus on axisymmetric data. More precisely we assume that in cylindrical coordinates (r,θ,z), the initial velocity field v^0 is given by

$$v^0(r,\theta,z) = v_r^0(r,z)e_r + v_z^0(r,z)e_z$$

where e_r stands for the unit (outer) radial vector, and e_z for the unit vertical vector.

An easy computation shows that the initial vorticity ω^0 then reduces to

$$\omega^0(r,\theta,z) = \omega_\theta^0(r,z)e_\theta \quad \text{with} \quad \omega_\theta^0 := \partial_z v_r - \partial_r v_z \quad \text{and} \quad e_\theta = e_z \times e_r.$$

It turns out that for axially symmetric flows, the quantity $r^{-1}\omega_\theta$ is transported by the flow, hence plays the same role as the vorticity in dimension two.

This fact has been used by M. Ukhovskiì and V. Yudovich in 1968 who stated global existence for axisymmetric initial vorticity ω^0 such that ω^0 and $r^{-1}\omega^0$ belong to $L^2 \cap L^\infty$ (see [3]).

Note that in terms or regularity in Sobolev spaces, these assumptions are stronger than those which are needed to have local well-posedness in dimension three. Indeed, on one hand, local existence holds true for any solenoidal v^0 in $H^s(\mathbf{R}^3)$ with $s > 5/2$; on the other hand $s > 7/2$ is required for having $r^{-1}\omega^0$ in L^∞ for all axisymmetric $v^0 \in H^s(\mathbf{R}^3)$. This gap has been filled in by T. Shirota and T. Yanagisawa [2] who proved global existence for axially symmetric initial velocity field in H^s with $s > 5/2$.

We aim at getting a global existence result as close as Yudovich's result in dimension two. More precisely, we would like a global result for ω^0 in a functional space which has the same scaling invariance as $L^\infty(\mathbf{R}^3)$. It may be shown that the set of bounded functions ω such that $r^{-1}\omega$ belongs to the *Lorentz space* $L^{3,1}(\mathbf{R}^3)$ (which may be defined by real interpolation as $L^{3,1}(\mathbf{R}^3) = (L^\infty(\mathbf{R}^3), L^1(\mathbf{R}^3))_{\frac{1}{3},1}$) has the same scaling as $L^\infty(\mathbf{R}^3)$. Now we formulate our main result.

Theorem 1. *Let Ω be either a smooth axially symmetric bounded domain of $\mathbf{R}^3$, or the whole $\mathbf{R}^3$. Let ω^0 be an axisymmetric function in $L^{3,1}(\Omega) \cap L^\infty(\Omega)$ such that $r^{-1}\omega^0 \in L^{3,1}(\Omega)$. Let v^0 be the axisymmetric solenoidal vector-field with vorticity $\omega^0 e_\theta$ given by the Biot–Savart law. Then the Euler equations* (E) *have a unique global solution with vorticity ω in $L^\infty_{loc}(\mathbf{R}; L^{3,1}(\Omega) \cap L^\infty(\Omega))$. Besides, $\|r^{-1}\omega(t)\|_{L^{3,1}}$ is independent of $t \in \mathbf{R}$.*

The proof of Theorem 1 relies on the following elementary three facts:

1. because $r^{-1}\omega_\theta$ is transported by the the flow of an *incompressible* vector-field, we have $\|r^{-1}\omega(t)\|_{L^{3,1}} = \|r^{-1}\omega^0\|_{L^{3,1}}$ for all $t \in \mathbf{R}$,

2. there exists a constant C depending only on Ω such that $\left\|r^{-1}v_r\right\|_{L^\infty} \leq C\|r^{-1}\omega\|_{L^{3,1}}$,

3. for axially symmetric flows, the equation for ω_θ reduces to

$$\partial_t \omega_\theta + v \cdot \nabla \omega_\theta = r^{-1} v_r \omega_\theta.$$

Uniqueness may be proved by adapting Yudovich's arguments.

In the axisymmetric framework, it may be shown that our assumptions are fulfilled whenever v^0 belongs to H^s for some $s > 5/2$. Actually, they are still satisfied in the limit case $s = 5/2$ if v^0 belongs to the *Besov space* $B^{\frac{5}{2}}_{2,1}$. On one hand, it is easy to show that a $B^{\frac{5}{2}}_{2,1}$ axisymmetric data generates a *local* $B^{\frac{5}{2}}_{2,1}$ solution and, by virtue of Theorem 1, a *global* unique solution in a larger space. To our knowledge, whether the $B^{\frac{5}{2}}_{2,1}$ regularity is conserved globally is an open problem.

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On the boundary control in a flat non-stationary flow with vortex singularities

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We present formula (2) (comp. [1]), for a flat symmetric w.r.t. x axis non-stationary divergence-free flow

$$\widetilde{\Omega}(0) \ni z(0) = x(0) + iy(0) \mapsto \\ \mapsto z(t) = x(t) + iy(t) \in \widetilde{\Omega}(t) = \mathbf{R}^2 \setminus (S \cup z_1(t) \cup \overline{z}_1(t)),$$

that circulates around a quadrangle $S(t)$. The quadrangle is parameterized by the angles $\beta_0(t)$ and $\beta_1(t)$ formed by the sides (rear and frontal respectively w.r.t. the coming flow) of the triangle $S(t) \cap \mathbf{R}^2$ with the x axis. The velocity $\vec{V}(t,z)$, which obeys the Euler equation

$$\frac{d\vec{V}(t,z(t))}{dt} = -\nabla_{x,y}\, p(t,z)\big|_{x=x(t),\; y=y(t)},$$

is potential outside unknown vortex centers $z_1 = x_1 + iy_1$ and $\overline{z}_1 = x_1 - iy_1$ with $y_1 > 0$. This means that $\vec{V}(t,z) = \nabla_{x,y} u(t;x,y)$.

Let $\Omega(t) = (\widetilde{\Omega}(t) \cap \mathbf{R}^2_+) \setminus l(t)$, where $l(t)$ is the cut along the level set of the potential $u(t;\cdot,\cdot)$ from the midpoint of the rear piece of boundary $S(t)$ to the unknown vortex center $z_1(t)$. An analitical function $w(t;\cdot,\cdot) = u(t;\cdot,\cdot) + iv(t;\cdot,\cdot)$ maps univalently $\Omega(t)$ to the domain $Q_{\sigma(t)}$ parameterized by a vector $\sigma(t) = (\sigma_1(t), \dots, \sigma_4(t))$, which is a solution of the following system

$$\sum_{1 \le k \le 4} c_{kl}(\sigma)\frac{d\sigma_k}{dt} + d_l(\sigma) = 0. \tag{1}$$

The domain $Q_{\sigma(t)}$ is actually a half-plane with a half-strip $\{\sigma_1(t) < u < \sigma_4(t),\ v < 0\}$ pasted up along the interval $\{\sigma_2(t) < u < \sigma_3(t),\ v = 0\}$.

Theorem. *The function* $z : t \mapsto z(t) = x(t) + iy(t)$ *of the center of an elementary fluid volume and of its velocity* $\vec{V}(t, x(t) + iy(t)) \stackrel{\text{def}}{=} \frac{d}{dt} z(t)$ *is given by the following explicit formula*:

$$z(t) = z(0) + \int_{w_0}^{w(t)} \exp\left\{A(\xi, \eta; \sigma(t), \beta(t))) + iB(\xi, \eta; \sigma(t), \beta(t)))\right\} d(\xi + i\eta)$$
$$w_0 = w(z(0)), \tag{2}$$

where the function $t \mapsto w(t) = u(t) + iv(t)$ *can be expressed by the solution of the following ODE system:* $du/dt = f(t, w, \sigma, \beta)$, $dv/dt = g(t, w, \sigma, \beta)$ *with explicitly given functions* f *and* g. *The formulas for* f *and* g *contain (for every* t*) functions* $A = A(\cdot, \cdot; \sigma(t), \beta(t)))$ *and* $B = B(\cdot, \cdot; \sigma(t), \beta(t)))$ *harmonic conjugate in the domain* $Q_{\sigma(t)} \ni w = u + iv$ *and its derivatives is the parameters* σ *and* $\beta = (\beta_0, \beta_1)$. *Moreover, for any* t *one has* $A(u, v; \sigma(t), \beta(t)) \to 0$ *as* $u + iv \to \infty$, *and the function* B *satisfies the following boundary condition: for* $v = \pm 0$

$$B(u, +0; \sigma(t), \beta(t)) = \begin{cases} 0, & \text{if} \quad u \in (-\infty, 0) \\ \beta_0(t), & \text{if} \quad u \in (0, \sigma_2(t)) \\ 0, & \text{if} \quad u \in (\sigma_3(t), \infty) \end{cases},$$

$$B(u, -0; \sigma(t), \beta(t)) = \begin{cases} \beta_1(t), & \text{if} \quad u \in (\sigma_1(t), \sigma_2(t)) \\ -\pi, & \text{if} \quad u \in (\sigma_3(t), \sigma_4(t)) \end{cases}$$

for $v < 0$

$$B(u, v; \sigma, \beta)\big|_{u=\sigma_1} = B(u, v; \sigma, \beta)\big|_{u=\sigma_4} + 2\pi,$$
$$\partial_u B(u, v; \sigma, \beta)\big|_{u=\sigma_1} = \partial_u B(u, v; \sigma, \beta)\big|_{u=\sigma_4}.$$

Each equation in (1) *is actually an orthogonal projection in* $L^2(\Omega)$ *of the Euler equation onto the corresponding divergent-free function. The coefficients* c_{kl} *and* d_l *can be computed by explicit formulas with the functions* A *and* B.

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Shock wave structures in thin film flows

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Shock waves arise in a wide variety of disciplines, for example gas dynamics, traffic flow, magnetohydrodynamics and thin film flow. They emerge as discontinuous solutions of hyperbolic wave equations of the form

$$\frac{\partial h(x,t)}{\partial t} + \frac{\partial f(h(x,t))}{\partial x} = 0.$$

When the flux function $f(h(x,t))$ is non-convex, it is well known that shock discontinuities are sensitive to regularisation of the hyperbolic equation, this can lead to new phenomena including the generation of non-classical shock waves. In the work presented here we analyze thin film flow systems and explore regularisations that models surface tension, curvature, surface tension gradient and variable gravity effects associated with flow along a curved solid substrate. The consideration of such effects introduces spatially dependent coefficients which can lead to the birth of a non-classical shock from a classical shock, termination of a non-classical shock in finite space and time, the generation of sonic shocks, disintegration of shocks into rarefraction fans among other types of wave interaction.

We also consider the case of a diffusive and dispersive regularisation and we use this to investigate the embryonic formation of a non-classical shock.

Homogeneous and isotropic statistical solutions of the Navier–Stokes equations

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We shall report on joint work with A. V. Fursikov and J. D. Kahl on the existence of homogeneous and isotropic statistical solutions of the 3D Navier–Stokes system.

First, we shall discuss how to obtain homogeneous and isotropic probability measures, supported by weak solutions of the Navier-Stokes system, by averaging over rotations the known homogeneous statistical solutions by M. I. Vishik and A. V. Fursikov.

We shall then discuss how to approximate (in the sense of convergence of characteristic functionals) any isotropic measure on a certain space of vector fields by isotropic measures supported by periodic vector fields and their rotations. This can be achieved without loss of uniqueness for the Galerkin system, allowing for the Galerkin approximations of homogeneous statistical Navier–Stokes solutions to be adopted to isotropic approximations. The Vishik–Fursikov construction of homogeneous measures then applies to produce homogeneous and isotropic probability measures, supported by weak solutions of the Navier–Stokes equations.

In both constructions, the restriction of the measures at $t = 0$ is well defined and coincides with the initial measure.

Some decompositions of Sobolev spaces and complex boundary-value problems

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Some nonstandart complex BVP's are connected with the following decomposition of Sobolev scale $W_p^m(G)$, where G is a bounded domain $G \subset \mathbb{C}_z^n \approx \mathbb{R}_{x,y}^{2n}$, $n \geq 1$, $m = 0, \pm 1, \ldots$, $p > 1$:

$$W_p^m = S_p^m + \nabla_{\bar{z}} W_{p,0}^{m+1}.$$

Here S_p^m is the subspace of complex solenoidal functions, $W_{p,0}^{m+1}$ is the Sobolev space of potentials with zero boundary values (in the weak sence for $m \leq 0$), $\nabla_{\bar{z}} = (\partial_{\bar{z}_1}, \ldots, \partial_{\bar{z}_n})$ — the Cauchy–Riemann gradient.

These decomposition implies the normal solvability (in the sense of Hausdorf) of the Dirichlet problems for the Cauchy-Riemann system

$$\nabla_{\bar{z}}\, p(z) = q(z), \qquad p(z)|_\Gamma = 0,$$

in the scales W_p^m and $\mathring{W}_p^m$.

Further we consider the scale

$$D_p^{m,r} = \left\{ u(z) : \; u(z) \in W_p^m \;\&\; \mathrm{div}_z u(z) \in W_p^{m-1+k} \right\},$$

where $k \geq 0$. For this scale we also have the analogy decomposition

$$D_p^{m,k} = S_p^m + \nabla_{\bar{z}} W_{p,0}^{m+1+k}.$$

This decomposition shows that $D_p^{m,k}/S_p^m \subset W_p^{m+k}$, i.e. the solenoidal factorization is the "smoothness" action. This fact gives the possibility to proof the normal Hausdorf solvability of the various BP for the equation of type

$$\nabla_{\bar{z}} \left(a(z)\, |\mathrm{div}_z u|^{p-2}\, \mathrm{div}_z u \right) = h(z), \quad p > 1.$$

The third decomposition is

$$W_p^m = \mathcal{O}_p^m + \mathrm{div}_z W_{p,0}^{m+1},$$

where $\mathcal{O}_p^m$ is the subspace of analytic functions and $W_{p,0}^{m+1}$ is the Sobolev space of functions, analytic boundary part of which is equal to zero.

Finally, we study the analytic BVP of variational type.

Conformal invariance in two-dimensional turbulence

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Simplicity of fundamental physical laws manifests itself in fundamental symmetries. While systems with an infinity of strongly interacting degrees of freedom (in particle physics and critical phenomena) are hard to describe, they often demonstrate symmetries, in particular scale invariance. In two dimensions (2d) locality often promotes scale invariance to a wider class of conformal transformations which allow for nonuniform re-scaling. Conformal invariance allows a thorough classification of universality classes of critical phenomena in 2d. Is there conformal invariance in 2d turbulence, a paradigmatic example of strongly-interacting non-equilibrium system?

We consider here 2d incompressible turbulent motion of a fluid, which represents an appropriate description of large-scale motions of the atmosphere and can be realized in different laboratory settings as well. As predicted by Kraichnan, stirring at a finite scale produces two turbulence cascades, with the formation of fine-scale vortical structures and large-scale velocity structures. In two dimensions, squared vorticity performs a direct cascade to small scales while kinetic energy flows from the injection length to large scales, opposite to the three-dimensional case. Note that the kinetic energy and all powers of vorticity are inviscid invariants of incompressible hydrodynamics. We focus here on the inverse cascade of energy which is known to have scale-invariant statistics with Kolmogorov-Kraichnan scaling.

Our goal was to find out whether scale invariance can be extended to conformal invariance at least for some properties of 2d turbulence. Under conformal transformations the lengths are re-scaled non-uniformly yet the angles between vectors are left unchanged (a useful property in navigation cartography where it is often more important to aim in the right direction than to know the distance). The novelty of our approach is that we analyze the inverse cascade by describing the large-scale statistics of the boundaries of vorticity clusters, i.e. large-scale zero-vorticity lines. In equilibrium critical phenomena, cluster boundaries in the continuous limit of vanishingly small lattice size were recently found to belong to a remarkable class of curves that can be mapped into Brownian walk (called

Stochastic Loewner Evolution or SLE curves). Such random curves have conformally invariant statistics.

Using numerical experiment, we have identified zero-vorticity lines as SLE curves. The statistics of vorticity clusters is remarkably close to that of critical percolation, one of the simplest universality classes of critical phenomena. These results represent a new step in the unification of 2d physics within the framework of conformal symmetry.

On variational solutions to the complete Navier–Stokes–Fourier system

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A mathematical theory of variational (weak) solutions for the complete Navier–Stokes–Fourier system of equations built up on the second law of thermodynamics is discussed. Applicability of the theory is illustrated on several practical issues:

- existence theory for large data and large time intervals;
- long time behaviour of solutions and stability of the static states;
- the Oberbeck–Boussinesq system as the incompressible limit.

The complete Navier–Stokes–Fourier system can be written in the form

$$\partial_t \varrho + \nabla \cdot (\varrho \vec{u}) = 0,$$

$$\partial_t (\varrho \vec{u}) + \nabla \cdot (\varrho \vec{u} \otimes \vec{u}) + \frac{1}{[Ma]^2} \nabla p(\varrho, \vartheta) =$$

$$= \nabla \cdot \mu \Big(\nabla \vec{u} + \nabla^t \vec{u} - \frac{2}{3} \nabla \cdot \vec{u}\; Id \Big) + \frac{1}{[Fr]^2} \varrho \nabla g,$$

$$\partial_t (\varrho s(\varrho, \vartheta)) + \nabla \cdot (\varrho s(\varrho, \vartheta) \vec{u}) - \nabla \cdot (\frac{\kappa}{\vartheta} \nabla \vartheta) =$$

$$= \frac{1}{\vartheta} \Big[\mu [Ma]^2 \Big(\nabla \vec{u} + \nabla^t \vec{u} - \frac{2}{3} \nabla \cdot \vec{u}\; Id \Big) : \nabla \vec{u} + \frac{\kappa |\nabla \vartheta|^2}{\vartheta} \Big],$$

$$\frac{d}{dt} \int \Big([Ma]^2 \frac{1}{2} \varrho |\vec{u}|^2 + \varrho e(\varrho, \vartheta) - \frac{[Ma]^2}{[Fr]^2} \varrho g \Big)\, dx = 0,$$

with the state variables $\varrho = \varrho(t,x)$ the density, $\vec{u} = \vec{u}(t,x)$ the fluid velocity, $\vartheta = \vartheta(t,x)$ the absolute temperature.

The thermodynamics functions - the pressure $p = p(\varrho, \vartheta)$, the specific entropy $s = s(\varrho, \vartheta)$, the specific internal energy $e = e(\varrho, \vartheta)$ - are interrelated through Gibbs' equation

$$\vartheta Ds = De + pD\Big(\frac{1}{\varrho}\Big).$$

The above system is considered on a bounded spatial domain $\Omega \subset R^3$, and supplemented with the conservative boundary conditions

$$\vec{u}|_{\partial\Omega} = 0, \ \nabla\vartheta \cdot \vec{n}|_{\partial\Omega} = 0.$$

To begin with, we shall show that the problem admits a weak (variational) solution for large initial data defined on an arbitrary large time interval $(0, T)$. The concept of variational solutions is based on replacing the entropy production balance by an inequality taking into account possible singularities that may contribute to the entropy production. Formal consistency of the problem is then saved by the total energy balance that holds even in the class of variational solutions. Such a theory is in the spirit of the approach of Leray [2] and Ladyzhenskaya [1] for the incompressible case, developed later by Lions [3] for the compressible barotropic model.

With the existence theory at hand, we are allowed to discuss the problem of the long-time behaviour of the solutions. For the above system, where the driving force is of potential type, we show that any solution tends to a static state, that means, a zero velocity stationary solution, determined in a unique way by the initial mass and the total energy that are conserved quantities. Furthermore, one can show that for a general and even time dependent driving force, there only two alternatives: either it is of potential type and then any solution converges to a static state, or the total energy goes to infinity with growing time.

The dimensionless parameters Ma, Fr denote the Mach and Froude number, respectively. Taking $Ma \approx \epsilon$, $Fr \approx \sqrt{\epsilon}$ we shall show that the corresponding solutions tend for $\epsilon \to 0$ to a solution of the incompressible Boussinesq system. This is to be understood as another evidence of the physical relevance of the underlying mathematical theory.

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Mathematical study of equatorial waves

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The aim of this talk is to present some recent progress on the mathematical study of geophysical fluids, joint with Laure Saint-Raymond. We will mainly consider in this talk the case of the oceans in the equatorial zone of the Earth, and study the so-called "betaplane model," presenting the results of [2] (see also [1] for a survey on various mathematical studies of geophysical fluids).

Let us denote by h the depth of the water, assuming that $h = 1 + \varepsilon\eta$ where ε is a small, adimensionalized number; the function η represents the variation of the depth of the fluid. We denote by u the horizontal components of the velocity field. The fluid is supposed to be homogeneous and to satisfy the hydrostatic assumption, and we assume that the motion is goverened by a three dimensional Navier–Stokes system with free surface. The main assumption (unjustified but commonly accepted among Physicists) is that the horizontal motion is independent of the vertical coordinate. Under the betaplane approximation, the Coriolis force is assumed to be linear in the latitude x_1, of amplitude $\beta\varepsilon^{-1}$, and the Froude number (ratio of the characteristic velocity of the fluid and the internal waves) is also of order ε. The system is then the following:

$$\left\{\begin{array}{c} \partial_t\eta + \dfrac{1}{\varepsilon}\text{div } ((1+\varepsilon\eta)u) = 0 \\ \partial_t\left((1+\varepsilon\eta)u\right) + \text{div } ((1+\varepsilon\eta)u\otimes u) + \\ +\dfrac{1}{\varepsilon}\beta x_1(1+\varepsilon\eta)u^\perp + \dfrac{1}{\varepsilon}(1+\varepsilon\eta)\nabla\eta - \nu\Delta u = 0. \end{array}\right.$$

We will not go into the justification of the various assumptions and adimensionalizations leading to that system (we refer for instance to the books of

A. Gill [4] or J. Pedlosky [5]). In particular the choice of the diffusion operator is dictated by the fact that the depth of the fluid is assumed to be a perturbation of a mean value (1 to simplify here): one would expect a diffusion operator of the type div $(h\nabla u)$ but in our context taking simply Δu does not change the asymptotics, and simplifies the study. Moreover we choose to set the equations in the domain $R \times T$ although such an assumption is not very physical. For the x_1 variable it turns out that all the waves are rapidly decaying far from the equator, so that it can be expected that boundary conditions would not interfere in the general behaviour of the fluid, but in the x_2 direction boundaries should of course be considered. That should be the object of some future work.

Our aim in this talk is to study the behaviour of the solutions as the parameter ε goes to zero.

The first question to address is that of the existence of solutions, uniformly bounded in ε. It turns out that it is not difficult to adapt the usual, compressible Navier–Stokes theory to prove the existence of bounded energy solutions to that system, satisfying

$$\frac{1}{2}\int (\eta^2 + (1+\varepsilon\eta)|u|^2)(t,x)dx + \int_0^t \int \nu|\nabla u|^2(t',x)dxdt' \leq C.$$

The question we are interested in now is to describe the asymptotic behaviour of those solutions as the parameter ε goes to zero. Denoting by $(\eta_\varepsilon, u_\varepsilon)$ any family of bounded solutions associated with initial data in L^2, the above bound shows that there exist $\eta \in L^\infty(R^+; L^2(R \times T))$ and $u \in L^\infty(R^+; L^2(R \times T)) \cap L^2(R^+, \dot{H}^1(R \times T))$ such that, up to extraction of a subsequence,

$$(\eta_\varepsilon, u_\varepsilon) \rightharpoonup (\eta, u) \text{ in w-}L^2_{loc}(R^+ \times R \times T).$$

Our aim is now twofold: first, to describe as precisely as possible the weak limit (η, u), by computing the evolution equation it satisfies. Second, to describe the type of convergence that actually takes place: is the convergence weak or strong, and if it is weak, can one describe precisely the defect of compactness in the sequence $(\eta_\varepsilon, u_\varepsilon)$?

It is easy to see that (η, u) belongs to the kernel of the penalization operator

$$L(\eta, u) \in L^2(R \times T) \mapsto (\nabla \cdot u, \beta x_1 u^\perp + \nabla\eta),$$

and though the projection onto Ker$L$ involves a pseudo-differential operator with a singularity at $x_1 = 0$, it is nevertheless possible to project the original system onto Ker L to look for an evolution equation on (η, u). Indeed once the equation has been projected onto $\operatorname{Ker} L$, it is no longer singular in ε and time compactness can be recovered. Taking limits in the nonlinear terms requires some care, but a weak compactness argument allows to prove that the limit equation on (η, u) is a linear, heat-type equation.

The next step consists in studying the defect of compactness of the sequence $(\eta_\varepsilon, u_\varepsilon)$. In order to do so the idea is to study the spectrum of L. One proves that L can be diagonalized, and its eigenvectors can be explictly computed (and they indeed decay rapidly far from the equator). We recover the various waves well-known of Physicists (namely Poincaré, Kelvin and Rossby waves), and the study of the resonances induced by those waves allow, in the spirit of S. Schochet's work, to derive a limit system and to study its well posedness. Once that has been done, Schochet's filtering method enables one to prove a strong, local in time convergence result of the filtered solutions $e^{\frac{tL}{\varepsilon}}(\eta_\varepsilon, u_\varepsilon)$ towards the solutions of that limit system.

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Quaternions and particle dynamics in the Euler fluid equations

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Quaternions have fallen in and out of fashion since their invention by Hamilton but have staged a revival through their usefulness in modern inertial navigation systems, robotics, and graphics which control or track rapidly moving objects undergoing three-axis rotations (Kuipers 1999, Hanson 2006). A quaternion can be constructed from a scalar s and a 3-vector r by forming the tetrad $q = [s,\, r]$ that is defined by $[s,\, r] = sI - r\cdot\boldsymbol{\sigma}$ where $r\cdot\boldsymbol{\sigma} = \sum_{i=1}^{3} r_i\sigma_i$ and σ_i are the Pauli spin matrices. They are associative but non-commutative.

Motivated by a desire to study Euler particle and vorticity dynamics in a geometric manner, my collaborators (Holm, Kerr, and Roulstone: http://arxiv.org/abs/nlin.CD/0512034) and I decided to apply quaternions to the $3D$ Euler equations for an incompressible fluid whose velocity and vorticity formulation is

$$\frac{Du}{Dt} = -\nabla p\,, \qquad \operatorname{div} u = 0 \qquad \frac{D\boldsymbol{\omega}}{Dt} = S\boldsymbol{\omega}\,. \tag{1}$$

The strain matrix is written as $S = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ and $\boldsymbol{\omega} = \operatorname{curl} u$ is the vorticity; D/Dt is the usual Lagrangian (material) derivative. Euler data is known to become rough very quickly so all manipulations are formal. A natural tetrad $\boldsymbol{\zeta}(\boldsymbol{x},\, t) = [\alpha,\, \boldsymbol{\chi}]$ is formed from the scalar growth rate $\alpha = \widehat{\boldsymbol{\omega}}\cdot S\widehat{\boldsymbol{\omega}}$ and the 3-vector rotation rate $\boldsymbol{\chi} = \widehat{\boldsymbol{\omega}} \times S\widehat{\boldsymbol{\omega}}$. The Lagrangian equation for the vorticity tetrad $\boldsymbol{\Omega} = [0,\, \boldsymbol{\omega}]$ and $\boldsymbol{\zeta}$ can then be written as ($\otimes$ stands for quaternionic multiplication)

$$\frac{D\boldsymbol{\Omega}}{Dt} = \boldsymbol{\zeta}\otimes\boldsymbol{\Omega}\,, \tag{2}$$

where $\boldsymbol{\zeta}$ is shown to obey

$$\frac{D\boldsymbol{\zeta}}{Dt} + \boldsymbol{\zeta}\otimes\boldsymbol{\zeta} + \boldsymbol{\zeta}_p = 0\,, \qquad\qquad \frac{D\boldsymbol{\zeta}_p}{Dt} = \boldsymbol{\zeta}\otimes\boldsymbol{\zeta}_p + \boldsymbol{\Pi}\,. \tag{3}$$

$\boldsymbol{\zeta}_p$ is defined in the same manner as $\boldsymbol{\zeta}$ but with S replaced by $P = \{p_{,ij}\}$, which is the Hessian matrix of the pressure. This appears as the price to be paid for the use of Ertel's Theorem. $\mathbf{\Pi}$ is a tetrad linear in $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}_p$ whose scalar coefficients, in principle, are determined by the Poisson pressure relation. From equations (2) and (3) two results are demonstrated:

1. At each point in space-time a fluid particle carries its own orthonormal co-ordinate system $(\widehat{\boldsymbol{\omega}}, \widehat{\boldsymbol{\chi}}, \widehat{\boldsymbol{\omega}} \times \widehat{\boldsymbol{\chi}})$ for which explicit equations for Lagrangian time derivatives of this frame are given: the corresponding Darboux vector is the particle rotation rate. The frame-equations are directly related to the Frenet–Serret relations of differential geometry;

2. A theorem on the direction of vorticity is proved involving $\|\boldsymbol{\chi}_p\|_\infty$. Although differing in detail because it specifically depends upon the Hessian P, this is in the same style as the direction of vorticity theorems expressed as variations on the Beale–Kato–Majda theorem (1984).

It also turns out that the Lagrangian-quaternionic format displayed here is applicable to the equations for ideal MHD. Two time-clocks and two tetrads $\boldsymbol{\zeta}^{\pm} = [\alpha^{\pm}, \boldsymbol{\chi}^{\pm}]$ arise naturally through the use of Elsasser variables.

Approximation of an atmospheric system attractor with the help of unstable periodic orbits

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In this study we are considering one of the simple atmospheric systems, namely barotropic model of atmospheric circulation on a rotating sphere. The equation of this system can be written as

$$\frac{\partial \Delta\psi}{\partial t} + J(\psi, \Delta\psi + l + H) = F - \alpha\Delta\psi + \mu\Delta^2\psi. \tag{1}$$

Here ψ is a dimensionless stream function, J and Δ are Jacobian and Laplacian operators, l is Coriolis parameter, H is orography, α and μ are friction coefficients. After appropriate Galerkin approximation instead of (1) we have a system of ordinary differential equation.

For our set of parameter values the finite dimensional system is chaotic and has an attractor. Numerical calculation shows that all but one Lyapunov exponents of the system are nonzero. In this case one can try to approximate the characteristics of the system attractor using unstable periodic orbits of the system.

If rewrite the evolution of the system using its resolving operator S the periodic orbits will be solutions of the nonlinear equation $S(T)(\psi_0) = \psi_0$ with respect to the initial position ψ_0 and period T. To solve this equation numerically (and find periodic solutions) we developed several numerical methods including direct damped Newton method (with additional phase condition), complete Gauss-Newton method for minimizing $|S(T)(\psi_0) - \psi_0|$ and quasi-Newton methods with LBFG-S and GMRES schemes. It was shown that damped Newton method performs better than others. As a result of numerical calculations we got a set of periodic orbits for our system.

Let S_{pn} be a periodic orbit of our system. The index n will denote the period of the orbit. Index p will enumerate orbits with the same period n. As it follows from (Ruelle, *J. Stat. Phys.*, 1999) the average over the system attractor can be approximated as

$$\int F(u)d\mu = \lim_{N\to\infty} \frac{\sum_{n=1}^{N}\sum_{p=1}^{P}\alpha_{np}\overline{F_{np}}}{\sum_{n=1}^{N}\sum_{p=1}^{P}\alpha_{np}}. \tag{2}$$

In the above formula F_{np} is the average of F for the p-th n-orbit and α_{np} is the measure of orbit instability ($\exp(\lambda_+^i)$ are unstable multiplicators of the orbit)

$$\alpha_{np} = n/\exp(\sum_i \lambda_+^i).$$

We used the above representation to approximate the average state of the system (i.e., $F(u) \equiv u$). Calculating numerically both sides of equation (2), we found that the equation (2) holds with reasonable accuracy. This can be due to the fact that the set of unstable periodic orbits is dense on the system attractor and, consequently, every statistical characteristics of our system can be calculated according to (2).

Stability, instability, and stability in deep-water surface waves

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Here we report experiments on permanent form gravity waves on deep water propagating in both one and two horizontal dimensions. We find that moderate amplitude, bi-periodic patterns are "stable" within the length of our wave basin. This result is surprising in light of classic instability results (the Benjamin–Feir instability) for deep-water waves. And large amplitude experiments do show evidence of what appears to be the Benjamin–Feir instability. However, recent numerical results (Fuhrman and Madsen, 2006) provide a different explanation. Our further experiments show that their explanation is correct and the patterns are indeed "stable." To explain the unexpected persistence of these patterns mathematically, we reconsider the stability of a uniform wavetrain using the nonlinear Schroedinger (NLS) equation modified to include linear damping. We prove that the presence of damping, no matter how small, stabilizes (with linear and nonlinear stability) the uniform wavetrain solution. The predicted evolutions are in excellent agreement with our experiments. These stability results are then extended to the case of a permanent form solution of coupled NLS equations that model wave patterns.

The damped and driven Navier–Stokes system on large domains

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We consider the damped and driven 2D Navier–Stokes system

$$\partial_t u + \sum_{i=1}^{2} u^i \partial_i u = \nu \Delta u - \mu u - \nabla p + f,$$
$$\operatorname{div} u = 0, \qquad u(0) = u_0,$$

in the two-dimensional periodic domain $\Omega = [0, L/\alpha] \times [0, L]$ with area $|\Omega| = L^2/\alpha$, $\alpha \le 1$. The damping term μu is the Rayleigh (or Ekman) friction which plays an important role in geophysical models of atmospheric and oceanic circulation.

We are interested in sharp estimates for the number of the degrees of freedom for this system (expressed in terms of the fractal dimension of the global attractor $\mathcal{A}$ and the number of determining modes and nodes) as both $\alpha \to 0^+$ and $\nu \to 0^+$.

Theorem. *The fractal dimension of the global attractor satisfies the estimate*

$$\dim_F \mathcal{A} \le 12 \frac{L^2 \|\mathrm{rot} f\|_\infty}{\alpha\mu\nu}.$$

For the Kolmogorov-type forcing $f = f_{\mathrm{Kolm}}$ the fractal dimension satisfies the lower bound

$$\dim_F \mathcal{A}_{\mathrm{Kolm}} \ge \mathrm{const} \frac{L^2 \|\mathrm{rot} f_{\mathrm{Kolm}}\|_\infty}{\alpha\mu\nu},$$

which shows that the estimate is sharp as both $\alpha \to 0^+$ and $\nu \to 0^+$.

The numbers of determining modes and nodes satisfy the bounds that are of the same order as the fractal dimension:

$$N_{\mathrm{modes}} \le \frac{2}{\pi^2} \frac{L^2 \|\mathrm{rot} f\|_\infty}{\alpha\mu\nu}, \qquad N_{\mathrm{nodes}} \le (68)^{1/2} \frac{L^2 \|\mathrm{rot} f\|_\infty}{\alpha\mu\nu}.$$

This is a joint work with E. S. Titi.

Numerical realization of the method of functional-analytic series for projecting on a stable manifold

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The problem of initial data projecting on a stable manifold for the equation

$$\begin{cases} \partial_t u(t,x) - \partial_{xx} u(t,x) - \alpha u(t,x) + f(u) = 0, & \alpha = \text{const} \\ u(t,0) = u(t,\pi) = 0, & t \geq 0 \\ u(0,x) = u_o(x), & 0 \leq x \leq \pi \end{cases} \tag{1}$$

is solved numerically by the method of functional-analytic series.

Let H_- be the stable subspace of the linear part of the equation (1) and $H_+ = H_-^{\perp}$. We search for the stable manifold $\mathcal{M}_-$ related with the equation (1) in some neighborhood of zero $\mathcal{O}(H)$. Let us denote $\mathcal{O}(H_-) = \mathcal{O}(H) \cap H_-$.

The method in question is based on the fact that there exists a continuous mapping $F : \mathcal{O}(H_-) \to H_+$ such that $F(0) = 0$, $F'(0) = 0$ which sets the stable manifold $\mathcal{M}_-$ by $\mathcal{M}_- = \{u_- + F(u_-),\ u_- \in \mathcal{O}(H_-)\}$.

Computational formulas are deduced by series expansion of u_- and $F(u_-)$ with respect to the orthonormal basis $\{e_1 k\}_{k=1}^{\infty}$ of the phase space H of the equation (1), where $e_1 k(x) = \sqrt{\dfrac{2}{\pi}} \sin kx$ are the eigenvectors of the linear operator $A = -\partial_{xx} - \alpha$.

It should be emphasized that the method of functional-analytic series is usually used for theoretical investigation of the existence and smoothness of stable manifolds. The question of its applicability to numerical calculations is significant because the method is efficient for solving problems which need numerous projecting on a stable manifold. The principal result of the report is practical realization of the method in question as well as examination of computational stability of the realization considered in the process of approximate projecting on stable manifold in the case of quadratic nonlinearity.

Problem of steady-state flow over a step in the shallow-water approximation

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The study is aimed at solving the problem of a steady-state fluid flow over a step in the shallow-water approximation. The shallow-water equations are an approximation of the Euler equations for invicsid fluid with a free surface in the gravity field and hence, inherit all restrictions adopted in deriving these initial equations, in particular the condition of simple connectedness of the region occupied by the fluid in the neighborhood of the step. In this work, taking into account this condition made it possible to find the new steady-state flow regimes. The limitations on the possible flows depending on the flow direction are considered. All flow regimes characterized by the ratio of the fluid depth to the step height and the flow direction is found. Analytical expressions for the limitations imposed on the hydrodynamic flow parameters are obtained for each flow regime. The classical shallow -water equations coincide to within the notation with compressible-gas equations, the Prandtl number now plays the role of the Mach number and, depending on its absolute value, the flow is called subsonic or supersonic. We obtained three flow regimes:

(1) subsonic flow ahead of the step and supersonic or subsonic flow behind the step,

(2) supersonic flow ahead of the step and supersonic or subsonic flow behind the step,

(3) supersonic flow ahead of the step and sonic flow behind the step.

Regimes analysis showed that there exist two kinds of non-uniqueness of the solution. The first kind is attributable to the existence of two solutions with the same parameters to the left (or to the right) of the step. The second kind is related to the fact that the equations admit the transition from supersonic to subsonic flow unlimited number of times both ahead of and behind the step. The non-uniqueness of both kinds is the drawback of the shallow-water model. An alternative way to approximate hydrodynamic equations in situations when classical shallow-water models fail is suggested.

Existence of solutions to stochastic Navier–Stokes equation

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We begin with a general theorem on existence of solutions of stochastic equations and apply it to the stochastic Navier–Stokes equation.

Let H be a real separable Hilbert space w.r.t. $(\cdot,\cdot)_H$ and W be such that $(H,\ W)$ be an abstract Wiener space. Let $e_1, e_2, \dots$ be an orthonormal basis in H that belongs to some dense subspace X of H and is such that $\forall x \in X$ the sequence of projections on $\mathrm{Span}\{e_1, e_2, \dots, e_n\}$ converges to x w.r.t. a norm $\|\cdot\|_X$ which is not weaker than $\|\cdot\|_H$. The space X is Banach w.r.t. $\|\cdot\|_X$. The space Y is the completion of X w.r.t. a norm $\|\cdot\|_Y \le \|\cdot\|_H$.

A Lyapunov function $V : X \mapsto \mathbb{R}$ is called a reducible Lyapunov function iff there is an orthogonal basis $e_1, e_2, \dots$ in H such that $\forall j,k,n \in \mathbb{N}$ and $\forall x \in X$

$$k > n \Rightarrow \partial_k V(\pi_n x) = 0 \qquad j > n \text{ or } \ k > n \Rightarrow \partial^2_{jk} V(\pi_n x) = q_k(\pi_n x)\delta_{jk},$$

where $q_k(\cdot) \ge 0$.

Theorem. *Let these conditions be satisfied.*
1. *The imbedding $X \subset H$ is compact.*
2. *For some reducible Lyapunov function V, some $c > 0$, and any $x \in Y^*$,*

$$\widehat{L}V(x) \le cV(x) - Q(x),$$

where $Q(x) \ge c_1\|x\|_X^2$ and $Q(x) \ge c_2\|x\|_H^{2+\alpha}$ for some $\alpha, c_1, c_2 > 0$.
3. *For some $C > 0$ and any $x, y \in X$,*

$$\|B(x)-B(y)\|_Y + \|\sigma(x)-\sigma(y)\|^2_{\mathcal{L}_{HS}(W;H)} \le\le C\,[Q(x)+Q(y)]^{1/2}\|x-y\|_H.$$

Then the equation

$$dx(t) = B(x(t))\,dt + \sigma(x(t))\,dw(t)$$

with $B(\cdot):\ X \mapsto Y$, $\sigma(\cdot):\ X \mapsto \mathcal{L}(W;H)$, $\mathrm{E}\,(w(s),e_i)_H(w(t),e_j)_H = \delta_{ij}\, s\wedge t$, and generator $\widehat{L}$ has a weak solution $x \in L^2_{\mathrm{loc}}(\mathbb{R}_+;H)\cap C_{\mathrm{loc}}(\mathbb{R}_+;Y)$.

Let U be an open bounded subset of $\mathbb{R}^n$ with smooth boundary ∂U,

$$X = \{u \in W^{2,1}(U)^{\times n} : u\big|_{\partial U} = 0 \ \text{ and } \ \partial_k u_k = 0\},$$
$$H = L^2(U)^{\times n}, \quad Y = (C_0^{(2)}(U)^{\times n})^*,$$
$$B(u) = \Delta u - u_k \partial_k u, \ \text{and} \ \sigma(u) = \sigma.$$

Then the theorem implies that the stochastic Navier–Stokes equation

$$du(t) = (\Delta u - u_k \partial_k u)\, dt + \sigma\, dw(t)$$

has a weak solution $u \in L^2_{\text{loc}}(\mathbb{R}_+; L^2(U)^{\times n}) \cap C_{\text{loc}}(\mathbb{R}_+; (C_0^{(2)}(U)^{\times n})^*)$.

Existence of a solution "in the large" for the 3d large–scale ocean dynamics equations

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For the 3D system of primitive equations describing large-scale ocean dynamics in the Cartesian coordinate system existence and uniqueness of a weak solution such that the norm $\|\widehat{\mathbf{u}}_x\|$, where $\widehat{\mathbf{u}} = (u_1, u_2)$, is continuous in time on $[0, T]$, is proved on an arbitrary time interval without any assumptions on smallness.

Namely, let Ω be a cylinder in R^3 of the form $\Omega = \Omega' \times [0, 1]$, where Ω' is a domain on the x, y-plane with a piecewise smooth boundary. The boundary $\partial\Omega$ is represented as $\partial\Omega = S_1 \cup S$, $S = \partial\Omega' \times [0, 1]$, so S_1 is upper and bottom surfaces of the cylinder and S is its lateral surface.

The system of PDEs

$$\begin{gathered} \widehat{\mathbf{u}}_t - \nu\Delta\widehat{\mathbf{u}} + l\widehat{\mathbf{u}} + \nabla' p + u_k \widehat{\mathbf{u}}_{x_k} = \mathbf{0}, \quad \frac{\partial p}{\partial x_3} = -g\rho, \\ \operatorname{div} \mathbf{u} = 0, \quad \rho_t - \operatorname{div}(\nu_1 \nabla\rho) + u_k \rho_{x_k} = 0 \end{gathered} \tag{1}$$

and the initial and boundary conditions

$$\widehat{\mathbf{u}}\cdot\mathbf{n} = \frac{\partial\widehat{\mathbf{u}}}{\partial n}\times\mathbf{n} = 0 \text{ on } S, \quad \frac{\partial\widehat{\mathbf{u}}}{\partial n} = \mathbf{0} \text{ on } S_1,$$

$$u_3 = 0 \text{ on } S_1, \quad \frac{\partial\rho}{\partial n} = 0 \text{ on } \partial\Omega, \tag{2}$$

$$\widehat{\mathbf{u}}(0,\,x) = \widehat{\mathbf{u}}_0(x),\ \int_0^1 \operatorname{div}'\widehat{\mathbf{u}}_0\,dz = 0, \quad \rho(x,0) = \rho_0(x)$$

is considered. Here $\mathbf{u} = (u_1, u_2, u_3)$, $\widehat{\mathbf{u}} = (u_1, u_2)$, $\nabla' q = (q_{x_1}, q_{x_2})$, $\widehat{\mathbf{v}}\times\mathbf{n} = v_1 n_2 - v_2 n_1$, $l\widehat{\mathbf{u}} = \omega(u_2, -u_1)$. Summation over repeating indices in products is assumed.

The following statement is proved.

Theorem. *Let* $\widehat{\mathbf{u}}_0 \in \mathbf{W}_2^2(\Omega)$, $\rho_0 \in W_2^2(\Omega)$ *and* $\int_0^1 \operatorname{div}'\widehat{\mathbf{u}}_0 dz = 0$. *Then for any* ν, $\nu_1 > 0$ *and arbitrary* $T > 0$ *the problem* (1), (2) *has in* Q_T *a unique weak solution such that* $\widehat{\mathbf{u}}^2$, $\widehat{\mathbf{u}}_z^2$, $\widehat{\mathbf{u}}_x$, $\widehat{\mathbf{u}}_{zx}^2$, $\widehat{\mathbf{u}}_t$, $\widehat{\mathbf{u}}_{tx} \in \mathbf{L}_2(Q_T)$, *and* ρ^2, ρ_x, ρ_{zx}, $\rho_{tx} \in L_2(Q_T)$. *The norm* $\|\widehat{\mathbf{u}}_x\|$ *is continuous in* t.

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On the problem of approximate projection onto the invariant manifolds of the Navier–Stokes equations

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For the two-dimentional Navier–Stokes equations, the problem of numerical projection on local-stable and local-unstable sets is examined. This local-stable and unstable sets are defined in the neighbourhood $\mathcal{O} \subset H$ of the stationary (periodic) solution u_a. We consider any trajectory $u_a + u \in \mathcal{O}$ and denote the resolving operator for the function u as $u(t) = S(t, u(0))$.

Denote by $\mathcal{W}^-(\mathcal{O})$ the stable invariant set of S in $\mathcal{O}$:

$$\mathcal{W}^-(\mathcal{O}) = \left\{u(0)\in\mathcal{O} : \exists u(t)\in\mathcal{O}, u(t) = S(t,u(0)), \quad \forall t \geq 0\right\},$$

and by $\mathcal{W}^+(\mathcal{O})$ the unstable invariant set:

$$\mathcal{W}^+(\mathcal{O}) = \{u(0) \in \mathcal{O} : \exists u(t) \in \mathcal{O}, u(0) = S(t, u(t)), \quad \forall t \geq 0\}.$$

Suppose, that for the operator S there exist projectors $P_+, P_- : H \to H$; the bounded linear operator L; and the continuous mapping $R(u) = S(u) - Lu$ that the following conditions are fulfilled:

$$\begin{gathered} P_+ + P_- = I, \quad \|P_+\| = \|P_-\| = 1; \\ L(P_+) = P_+H, \quad L(P_-H) \subset P_-H; \\ \|Lv\| \geq \mu_+\|v\|, \forall v \in P_+H, \mu_+ > 1; \\ \|Lw\| \leq \mu_-\|w\|, \forall w \in P_-H, \mu_- < 1; \\ \|R(u_1) - R(u_2)\| < \theta\left(\max\{\|u_1\|, \|u_2\|\}\right)\|u_1 - u_2\|, \forall u_i \in H \end{gathered}$$

where $\theta(\cdot)$ is a continuous positive nondecreasing function and $\theta(0) = 0$. Then the stable set is found as $\mathcal{W}^-(\mathcal{O}) = \{P_-u + f(P_-u),\, u \in \mathcal{O}\}$, where the mapping f is a solution of functional equation:

$$P_+S(w + f(w)) = f\big(P_-S(w + f(w))\big).$$

The unstable set is found in the form of $\mathcal{W}^+(\mathcal{O}) = \{P_+u + g(P_+u),\, u \in \mathcal{O}\}$, where g is a solution of equation:

$$P_-S(v + g(v)) = g\big(P_+S(v + g(v))\big).$$

In this work methods of solving the given equations are constructed and proved. Generalization on trajectory of sadle-type have been also obtained. Results for a neighbourhood of stable point and for a neibourhood of trajectories are shown.

Authors respects prof A. V. Fursikov for helpful discussion.

On the derivation of Fourier's law for coupled anharmonic oscillators

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We study the Hamiltonian system made of weakly coupled anharmonic oscillators arranged on a three dimensional lattice $\mathbb{Z}_{2N} \times \mathbb{Z}^2$, and subjected to a stochastic forcing mimicking heat baths of temperatures T_1 and T_2 on the hyperplanes at 0 and N. We introduce a truncation of the Hopf equations describing the stationary state of the system which leads to a nonlinear equation for the two-point stationary correlation functions. We prove that these equations have a unique solution which, for N large, is approximately a local equlibrium state satisfying Fourier law that relates the heat current to a local temperature gradient. The temperature exhibits a nonlinear profile.

Large time existence results for the water-waves equations and asymptotics

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Many of the interesting physical phenomena one can observe in the water-waves dynamics occur for quite large times. For instance, the well-known Korteweg-de Vries (KdV) approximation differs significatively from the simple $1D$-wave equation over large time scales only. In this particular example, a large time existence result for the water-waves equations exists, in the sense that it has been proven by Craig [1] and Schneider-Wayne [3] that there exists an exact solution to the water-waves equations close to the KdV approximation, and over the physically relevant time-scale.

However, the general well-posedness results of [4] and [2] do not provide estimates on the existence time of the solution. Such a result is an essential step towards the rigorous justification of the $2DH$ asymptotic approximations, such as the Kadomtsev-Petviashvilii (KP) equations for instance.

In this talk, we will adress the question of finding large time existence results for the water-waves equations. After a nondimensionalization of the equations, one can exhibit several dimensionless parameters which have both a physical meaning and a mathematical role. Namely, if a denotes the order of magnitude of the perturbation of the free surface, λ the order of magnitude of its wavelenth and h the mean depth, these parameters are

$$\mu = h^2/\lambda^2 \quad \text{and} \quad \epsilon = a/h,$$

the first when being called *shallowness parameter*. (Note that things are slightly different in nonisotropic regimes such as the weakly-transverse waves leading to the KP approximation).
We will discuss the influence of these two parameters on the existence time of the solutions, and focus particularly on the case when these parameters are small (they are also equal in the so-called long-wave regime).

We will then comment more specifically on the KP approximation which can be derived formally from the water-wave equations in the case of weakly transverse long-waves. We will rigorously derive a class of Boussinesq-like systems in this physical setting and show that the KP approximation can be rigorously justified from these systems. This latter step requires however strong and unphysical zero-mass and zero-momentum assumptions on the initial perturbations, because of the lack of regularity in time of the flow associated to the KP equation. The class of systems introduced above has therefore a much wider range of validity since it does not require such unrealistic assumptions. We also show that its dispersive properties are far better and that, when the zero-mass assumption are satisfied, it converges toward an exact solution of the water-wave equations with a better rate than the KP approximation.

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Existence of global solutions to a system of nonlinear dispersive equations

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We investigate the nonlinear dispersive effect of the Green–Naghdi (GN) equations

$$\eta_t + \big((1+\eta)u\big)_x = 0,$$
$$u_t + uu_x + \eta_x = \frac{1}{1+\eta}\partial_x\big[(1+\eta)^3(u_{xt} + uu_{xx} - u_x^2)\big],$$

as second order, shallow water approximations to the two-dimensional full water wave problem. Because the GN system was derived under the assumption of long wave-length l compared with the depth of water h without restrictions on the wave amplitude, it contains nonlinearly dispersive terms and provides a good example to show nonlinearly dispersive effect on the dynamics of the water wave problem.

Using a priori energy estimates and the nonlinear perturbation theory for semi-groups, we demonstrate that the GN system is locally well-posedness. We choose proper energies that are uniformly bounded with respect to $\epsilon = h/l$ for any ϵ sufficiently small. This technique enables us to estimate how long the GN system remains a good approximation to the full water wave problem.

In addition, we demonstrate that the GN system possesses some solutions that remain in a neighborhood of certain bounded, oscillating functions using its Hamiltonian structure and comparison methods. This fact demonstrates the nonlinear dispersion effect on the existence of global solutions to the GN system as a contrast to the dispersionless, first order shallow water approximations.

Furthermore, using the above results we discuss the issues of stability of solitary waves. Because the Hamiltonian structure of the GN system does not provide a Lyapunov functional that could be used for nonlinear stability analysis of its solitary waves directly, one may consider linear stability analysis first. Then we use a properly chosen function space to show that the small amplitude solitary waves are nonlinear stable relative to a family of global solutions of the GN system based on the linear stability analysis.

Some numerical results will also be shown to illustrate solutions of the physical models involved.

Curtain coating in microfluidics and the phenomenon of nonlocality in dynamic wetting

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One of the central issues in the physics of capillarity is the question of whether or not dynamic wetting, i.e., the process of spreading of a liquid over a solid surface, is a local phenomenon whose characteristics depend only on the speed at which the three-phase contact line moves across the solid substrate and the material parameters of the contacting media or is it nonlocal, i.e., dependent also on the flow field/geometry in the vicinity of the contact line. A flow configuration that offers sufficient flexibility to clarify this issue is the so-called "curtain coating."

In the study, curtain coating on a length scale typical of microfluidics is investigated theoretically in the framework of an earlier developed theory where dynamic wetting is treated as essentially a process of formation of a new liquid-solid interface. The results demonstrate that the actual dynamic contact angle between the free surface and the solid boundary depends not only on the wetting speed and material constants of the contacting media, as in the so-called "slip models," but also on the flow field/geometry in the vicinity of the moving contact line. In other words, for the same wetting speed the dynamic contact angle can be varied by manipulating the flow conditions. This outcome is consistent with the conclusions drawn earlier from macroscopic experiments.

Hydrodynamic limit of the Boltzmann equation

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In his sixth problem, Hilbert asked for a full mathematical justification of fluid mechanics equations starting from particle systems. If we take the Boltzmann equation as a starting point, this problem can be stated as an asymptotic problem. Namely, starting from the Boltzmann equation, can we derive fluid mechanics equations and in which regime?

From a physical point of view, we expect that a gas can be described by a fluid equation when the mean free path (Knudsen number) goes to zero.

A program in this direction was initiated by Bardos, Golse and Levermore who, using the the renormalized solutions to the Boltzmann equation constructed by DiPerna and Lions, set an asymptotic regime where one can derive different fluid equations (and in particular incompressible models) depending on the chosen scaling. Many restrictions and assumptions were made one the sequence of solutions to give a rigorous result.

During the last two decades this problem got a lot of interest. In this talk, we present some of the most recent results concerning these (rigorous) derivations. In particular we will present some results for soft potentials.

Finite element approach to the plane stationary flow with free boundary

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Let $\Omega \subset R^2$ be a bounded domain occupied by a fluid.

$$\Omega = \{0 < x_1 < l,\ 0 < x_2 < \varphi(x_1)\},$$

where φ is an unknown smooth function such that the angles between lines $\{0 < x_1 < l,\ x_2 = \varphi(x_1)\}$ and $\{x_1 = 0\}$, $\{x_1 = l\}$ are $\frac{\pi}{2}$. $\partial\Omega = \Gamma = \Gamma_1 \cup \Gamma_2$,

$$\begin{aligned}\Gamma_1 &= \{x_1 = 0,\ 0 \le x_2 \le \varphi(0)\} \cup \{x_1 = l,\ 0 \le x_2 \le \varphi(l)\}\\ &\quad \cup \{0 < x_1 < l,\ x_2 = 0\},\\ \Gamma_2 &= \{0 < x_1 < l,\ x_2 = \varphi(x_1)\}\end{aligned}$$

Γ_1 is a fixed part of the boundary, Γ_2 is a free boundary.

The velocity $\bar{v}$ and the pressure p satisfy to the boundary value problem

$$\begin{gathered}-\nu\Delta\bar{v} + (\bar{v}\cdot\nabla)\bar{v} + \nabla p = f, \quad \operatorname{div}\bar{v} = 0 \quad \text{in} \ \ \Omega,\\ \bar{v}|_{\Gamma_1} = 0, \quad \bar{v}\cdot\bar{n}|_{\Gamma_2} = 0, \quad T(\bar{v},p)\bar{n}|_{\Gamma_2} = \sigma H\bar{n},\end{gathered} \tag{1}$$

where $\bar{f}$ is a given vector function, ν and σ are positive constants, $\bar{n}$ is the unit outward normal to the Γ_2, $T(\bar{v},p)$ is the stress tensor, H is the curvature of Γ_2. The volume of the fluid is fixed by the condition

$$\int\limits_0^l \varphi(x_1)\,dx_1 = V.$$

So the triple $(\bar{v},p,\varphi)$ is unknown.

The solution $(\bar{v},p,\varphi)$ to the problem (1) is found by a successive approximation method according to the approach of V. A. Solonnikov and V. V. Pukhnachev. The $k+1$ approximation $(\bar{v}^{(k+1)},p^{(k+1)})$ is the solution to the Navier–Stokes equations in the domain $\Omega^{(k)}$ with known fixed boundary $\Gamma^{(k)} = \Gamma_1^{(k)} \cup \Gamma_2^{(k)}$,

$$\begin{aligned}\Gamma_1^{(k)} &= \{x_1 = 0,\ 0 \le x_2 \le \varphi^{(k)}(0)\} \cup \{x_1 = l,\ 0 \le x_2 \le \varphi^{(k)}(l)\}\cup\\ &\quad \cup\{0 < x_1 < l,\ x_2 = 0\}, \qquad \Gamma_2^{(k)} = \{0 < x_1 < l,\ x_2 = \varphi^{(k)}(x_1)\}\end{aligned}$$

$$
\begin{aligned}
& -\nu\Delta\bar{v}^{(k+1)} + (\bar{v}^{(k+1)}\cdot\nabla)\bar{v}^{(k+1)} + \nabla p^{(k+1)} = f, \\
& \operatorname{div}\bar{v}^{(k+1)} = 0 \quad \text{in} \;\; \Omega^{(k)}, \\
& \bar{v}^{(k+1)}|_{\Gamma_1^{(k)}} = 0, \quad \bar{v}^{(k+1)}\cdot\bar{n}^{(k)}|_{\Gamma_2^{(k)}} = 0, \\
& T\left(\bar{v}^{(k+1)}, p^{(k+1)}\right)\bar{n}^{(k)}\cdot\bar{\tau}^{(k)}|_{\Gamma_2^{(k)}} = 0,
\end{aligned}
\tag{2}
$$

where $\bar{n}^{(k)}$ and $\bar{\tau}^{(k)}$ are the normal and tangential vectors to the $\Gamma_2^{(k)}$ accordingly. $\varphi^{(k+1)}$ is the solution to the ordinary differential equation

$$
(\varphi^{(k+1)})'' = H^{(k+1)}\left[1 + (\varphi^{(k+1)})'^2\right]^{\frac{3}{2}},
$$

where

$$
H^{(k+1)} = \frac{1}{\sigma}T\left(\bar{v}^{(k+1)}, p^{(k+1)}\right)\bar{n}^{(k)}\cdot\bar{n}^{(k)}|_{\Gamma_2^{(k)}}.
$$

The solution of the nonlinear problem (2) is found as a limit of a sequence of solutions to the linear problems

$$
\begin{aligned}
& -\nu\Delta\bar{u} + \nabla q = g, \quad \operatorname{div}\bar{u} = 0 \quad \text{in} \;\; \Omega^{(k)}, \\
& \bar{u}|_{\Gamma_1^{(k)}} = 0, \quad \bar{u}\cdot\bar{n}^{(k)}|_{\Gamma_2^{(k)}} = 0, \\
& T(\bar{u}, q)\,\bar{n}^{(k)}\cdot\bar{\tau}^{(k)}|_{\Gamma_2^{(k)}} = 0.
\end{aligned}
\tag{3}
$$

The finite element scheme to the problem (3) is proposed.

Incorrectness in inverse problems of hydromechanics

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High-speed flows of liquid or gas are related to the large Reynolds numbers and a thin boundary-layer on the rigid body surface. Therefore the model of ideal liquid can be used. Knowing the body shape and neglecting the boundary-layer thickness, the velocity and pressure distributions in the flow can be easily calculated. More complicated problem arises when the body shape is unknown and some additional conditions are used for its definition. For example, the pressure is assumed to be constant over the surface of a supercavity. Such and more general problem with the prescribed pressure distribution over the unknown part of the shape is an inverse and very complicated one. In particular, it is difficult to analyze the correctness of the setting of a problem, i.e. existence, uniqueness and stability of the solution.

The slender body theory [1] and the first approximation equation [2, 3] give an opportunity to obtain very simple solution of the steady axisymmetric problem with a free boundary. This equation yields the existence and uniqueness of solution at arbitrary values of the cavitation and Froude numbers. According to the steadiness principle, a small change in the parameters determining the solution, ought to cause small changes in this solution. It was shown [3,4] that in some cases this principle is not valid, i.e. some solutions are unstable and the appropriate flows cannot be realized. For example, the supercavity flows of imponderable liquid are incorrect for a cone with negative cavitation numbers. For cavitators with the negative derivative of radius at the point of cavity origin some negative cavitation numbers are possible. Probably, a similar situation takes place in the 2D case too. In a liquid with gravity there are some restrictions on the values of the cavitati on and Froude numbers, [2]. The examples of such incorrectness are found in the unsteady supercavity flows, in the liquid with capillarity [3,4] and in the case of partial cavitation.

Negative pressure gradients at the body surface are necessary to avoid the separation and cavitation [5]. This brings up the question: can flow pressure gradient remain negative over the whole body surface? The correctness of this inverse problem is investigated with the use of first approximation equation and exact solutions of the Euler equations.

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Rigorous bounds on the Nusselt number

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We are interested in the transport of heat through a layer of viscous fluid which is heated from below and cooled from above. Two mechanisms are at work: Heat is transported by simple diffusion and by advection through the flow. The transport by advection is triggered by buoyancy (hotter parts have lower density) but is hindered by the no–slip boundary condition for the fluid velocity at the bottom and top surfaces.

After the Boussinesq approximation and neglecting inertia, the equations contain a single dimensionless parameter, the Rayleigh number Ra. It measures the relative strength of advection with respect to diffusion. For $Ra \gg 1$, the flow is aperiodic and the heat transport is mediated by plumes. As a consequence, the horizontally averaged temperature displays boundary layers.

Inspired by the work of Constantin and Doering, we are interested in rigorous bounds on the average heat transport (the Nusselt number Nu) in terms of Ra. By PDE methods, Constantin and Doering prove

$$Nu \lesssim Ra^{1/3} \log^{2/3} Ra.$$

We use the conceptually intriguing method of the background (temperature) field, introduced by Hopf for the Navier–Stokes equation and used by Teman et. al. for the Kuramoto–Sivashinski equation. We propose a background temperature field with *non–monotone* boundary layers; direct numerical simulations show an average temperature field with the same qualitative behavior. We obtain the slightly improved bound

$$Nu \lesssim Ra^{1/3} \log^{1/3} Ra.$$

The crucial ingredient is a maximal regularity statement for the Stokes operator in suitably weighted L^2–spaces.

This is joint work with Charles Doering and Maria Reznikoff.

New 2-nd-accuracy-order methods of solving the stationary axisymmetrical Navier–Stokes problem in spherical layers; investigations of the basic spherical Couette flows

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New finite-element (FE) realizations of the iterative methods with the boundary conditions (BC) splitting were developed in [1] to solve the axisymmetrical first boundary value problem in a spherical layer for the stationary Stokes system, as well as for the singulary perturbed Stokes-type system. The FE realizations of the sort possess high convergence rates and the second order of accuracy in the max-norm over the whole layer. For the Stokes system, the said iterative method, at a differential level, was advanced and grounded in [2] without restriction by the axisymmetric case. It appeared that the efficiency of the numerical method for the Stokes system in [1] drops with the increasing thickness of the spherical layer. With regard to this, a similar numerical method (algorithmically even simpler) was constructed in [3] and proved to be more effective for the spherical layers with sufficiently large relative clearances.

On the basis of the developed methods for the Stokes system and of the simplest method of successive approximations, one created new numerical iterative methods with the BC splitting to compute stationary axisymmetrical flows of viscous incompressible fluid between concentric spheres [4]. The methods, now for the nonlinear problem, also secure the second order of accuracy in the max-norm both for velocity and for pressure. They converge at not large Reynolds numbers. However, as comparisons with some data from the natural experiment showed, in the case of the thin spherical layers, they converge virtually in the domain of subcritical values of the Reynolds numbers (up to the first loss of stability).

Resorting to the latest methods for the Navier–Stokes system, the authors conducted numerical investigations on meshes with high resolution and carried out classification (by the type of the configurations of the lines of the current function level in a meridional plain) of the basic spherical Couette flows at various regimes of the rotation of the boundary spheres and for a fairly wide range of the clearances of the spherical layers.

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Solvability of stationary boundary-value problem for compressible Navier–Stokes equations

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The compactness properties of stationary solutions to compressible Navier–Stokes equations are investigated in three dimensions. The existence of generalized solutions is established. Assume that a flow of compressible Newtonian fluid in bounded domain $\Omega \subset \mathbb{R}^3$ is governed by the equations

$$\begin{gathered} \Omega: \ \operatorname{div}(\rho \otimes \mathbf{u}) + \nabla p(\rho) - \rho \mathbf{f} = \operatorname{div}\Pi, \quad \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial\Omega: \ \mathbf{u} = 0, \quad \int_\Omega \rho\, dx = M, \end{gathered} \tag{1}$$

where the viscous stress tensor is defined by

$$\Pi = \nu \left(\nabla \mathbf{u} + \nabla \mathbf{u}^\top - \frac{2}{3} \operatorname{div} \mathbf{u} \right) + \xi \operatorname{div} \mathbf{u}\, \mathbf{I} \ ,$$

ν, ξ are viscous coefficients, $\dfrac{4}{3}\,\nu + \xi > 0$, and $\mathbf{f} : \Omega \mapsto \mathbb{R}^3$ is a given continuous vector field. We suppose that the flow is barotropic and $p(\rho) = \rho^\gamma$. Recall that $\gamma = 5/3$ for monoatomic gases and $\gamma = 7/5$ for diatomic gases. The existence of solutions to general initial-boundary value problems for compressible Navier–Stokes was proved by Lions in and by Feireisl, Matušů-Nečasová, Petzeltová, Straškraba for all $\gamma > 9/5$. These results were essentially improved by Feireisl, Novotný, and Petzeltová, who proved the existence of generalized solutions for all $\gamma > 3/2$. The question of solvability of compressible Navier–Stokes equations for $\gamma < 3/2$ is still an open question. The main result of the work is the following

Theorem 1. *For any* $\mathbf{f} \in C(\Omega)$, $M \geq 0$ *and* $\gamma \geq 4/3$, *there is a generalized solution* $\mathbf{u} \in H^{1,2}_0(\Omega)$, $\rho \in L^\gamma(\Omega)$ *to problem* (1) *so that for all* $\varepsilon > 0$,

$$\| d^{1/2+\varepsilon} \rho |\mathbf{u}|^2 \|_{L^1(\Omega)} + \|p\|_{L^1(\Omega)} + \|\mathbf{u}\|_{H^{1,2}(\Omega)} \leq c(\mathbf{f}, \gamma, \Omega, M, \varepsilon).$$

Here $d(x) = \operatorname{dist}(x, \partial\Omega)$.

Spectral properties of the linear steady-state equations for viscous compressible fluid

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Let $(\overline{u}_1(x), \overline{u}_2(x), \overline{\nu}(x))$ be the steady-state solution of the system for the viscous compressible fluid, where $u_i(x),\ i = 1, 2$, is velocity vector field written in Lagrange coordinates and $\nu(x)$ is the specific volume. We make the linearization of this system on the mentioned solution and consider the following spectral problem

$$\frac{1}{\overline{\nu}(x)}\Delta u_i(x)+a_{i1}(x)\partial_{x_1}\nu(x)+a_{i2}(x)\partial_{x_2}\nu(x)-\lambda u_i = 0,\ \mathrm{div}u(x)-\lambda\nu(x) = 0,$$

$$a_{ii}(x) = \frac{1}{\overline{\nu}^2(x)} \cdot \frac{\partial \overline{u}_i(x)}{\partial x_i} + p'(\overline{\nu}),\ a_{ij}(x) = \frac{1}{\overline{\nu}^2(x)} \cdot \frac{\partial \overline{u}_i(x)}{\partial x_j},\ i = 1, 2,$$

where $p \in C^1(0, \infty)$ is the pressure. All functions in this equation are periodic in x. Let us rewrite this system in abstract form $(A(x, D) - \lambda E)U(x) = 0$, where $A : H \to H$, $H = L_2(T) \times L_2(T) \times H^1(T)$ and $D(A(x, D)) = H^2(T) \times H^2(T) \times H^1(T)$, $R(A(x, D)) \subset L_2(T) \times L_2(T) \times H^1(T)$. For λ belonging to the resolvent set, the operator $(A(x, D)-\lambda E)^{-1}$ is defined on H.

Definition 1 (see [1]). A closed operator $A(x, D) : H \to H$ is called sectorial, if there exists $\varphi \in \left(\frac{\pi}{2}, \pi\right)$ and $M \geq 1$ thats for some $a \in \mathbb{R}$ sector $S_{a,\varphi} = \{\lambda : 0 \leq |\arg(\lambda - a)| \leq \varphi,\ \lambda \neq a\}$ lies in resolvent set of operator $A(x, D)$ and $\|(A - \lambda E)^{-1}\| \leq \dfrac{M}{|\lambda - a|}$ for $\lambda \in S_{a,\varphi}$.

Theorem 1. *The operator $A(x, D)$ is sectorial and it spectrum is discret.*

Theorem 1 has many applications. For instance it can be used for stabilization of the evolution equations viscous compressible liquid with help of method created in [2].

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The eigenfunctions of the curl and Stokes operators and some classes of explicit solutions of the Navier–Stokes equations

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The relations between eigenvalues and eigenfunctions of the curl operator R and the Stokes operator S are indicated in two cases:

1) in a cube Q with periodic boundary conditions,

2) in a ball B with Dirichlet boundary conditions $\mathbf{v}\big|_{\partial B} = 0$ for S operator and $\mathbf{n}\cdot\mathbf{u}\big|_{\partial B} = 0$ for R operator, where $\mathbf{n}$ is a unit normal vector on ∂B.

The multiplicity of every nonzero eigenvalues λ of the curl operator are finite but the multiplicity of zero eigenvalue is infinite.

To each two eigenvalues $\pm\lambda$ of the curl operator corresponds the eigenvalue $\nu\lambda^2$ of the Stokes operator and inversely to the eigenvalue μ of the Stokes operator correspond two eigenvalues $\pm\sqrt{\dfrac{\mu}{\nu}}$ of the curl operator.

The eigenfunctions $\mathbf{u}^{\pm}$ and $\mathbf{v}$ of these operators R and S are calculated explicitly and we've proved that $\mathbf{v} = \mathbf{u}^{+} + \mathbf{u}^{-}$.

The functional vector space $\mathbf{L}_2(Q, 2\pi)$ may be decomposed into the direct sum of the curl's eigen-subspaces. The Curl'equation:

$$\operatorname{curl}\mathbf{u} + \lambda\mathbf{u} = \mathbf{f} \quad \text{in} \quad Q$$

and Stokes' equations with parameter λ:

$$-\nu(\triangle\mathbf{v} + \lambda^2\mathbf{v}) + \nabla p = \mathbf{f}, \quad \operatorname{div}\mathbf{v} = 0,$$

are solved. For any complex number λ the conditions of solvability and exact spaces for solutions are indicated. The results are useful in studying the Cauchy problem for the Navier–Stokes system with initial conditions in the class of periodic functions. We reduce this problem to the the Cauchy problem for the system of first order (ordinary) differential equations in Hilbert spaces. It may be solved by method of Galerkin's approximations. Local solutions exist but we are interested by the existence of global solutions and their bifurcation. We have written PC program to perform computing experiment. Moreover some families of explicit solutions of the (nonlinear) Navier–Stokes system are found.

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Properties of solutions of Navier–Stokes equations

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Let ε be a small positive parameter and (u,p) be a Hopf's solution of the initial-boundary value problem for unsteady Navier–Stokes equations

$$u'_t - \nu\,\Delta u + u\cdot\nabla u + \nabla p = F_\varepsilon \quad \text{in} \quad \Omega\times(0,T),$$

$$\operatorname{div} u = 0 \quad \text{in} \quad \Omega\times(0,T), \tag{1}$$

$$u\big|_{t=0} = 0 \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega\times(0,T),$$

where $F_\varepsilon = F(t,x,x/\varepsilon)$, $F(t,x,y) \in L^2(0,T;L^2(\Omega;L^\infty_{per}(Y)/R)^n)$, $\Omega \subset R^n$ is a bounded domain with a smooth boundary, T is a positive number, and $2 \le n \le 4$. Here, a subscript *per* means 1-periodicity with respect to $y \in R^n$ and $Y = [0,1]^n$ is a periodicity cell. Thus, by definition $F(t,x,y)$ is 1-periodic in y, $\int_Y F(t,x,y)\,dy = 0$ for a. e. $(t,x) \in (0,T)\times\Omega$, and the restriction of $F(t,x,y)$ to Y is an element of $L^2(0,T;L^2(\Omega;L^\infty(Y))^n)$.

Theorem. *Let $\nabla_x F \in L^1(0,T;L^2(\Omega;L^\infty_{per}(Y)/R)^{n\times n})$ and (u,p) is a solution of problem (1). Then, there are positive ε_0 and ν_0 such that*

$$\|u\|^2_{L^\infty(0,T;L^2(\Omega)^n)} + \nu\,\|\nabla u\|^2_{L^2(0,T;L^2(\Omega)^{n\times n})} \le C\,(\varepsilon^2 + \varepsilon^2\,\nu^{-1})$$

and

$$\|p\|_{W^{-1,\infty}(0,T;L^2(\Omega)/R)} \le C\,(\varepsilon + \varepsilon^2\,\nu^{-1-n/4}),$$

where C is a constant independent of ε and ν whenever $0 < \varepsilon \le \varepsilon_0$ and $0 < \nu \le \nu_0$.

Methods of homogenization and [1–3] are used to prove of the theorem. For example, the estimate for u is actual when $\varepsilon^2\,\nu^{-1} \to 0$ as $\varepsilon \to 0$. According to [3], in the case of linearized equations, an approximation of u contains rapidly oscillating terms if $\varepsilon^2\,\nu^{-1}$ tends to a constant as $\varepsilon \to 0$.

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Questions regarding the 2D quasi-geostrophic equations

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Consideration will be given to the long time behavior of solutions to the dissipative 2D quasi-geostrophic flow with sub-critical powers. This flow is described by the nonlinear scalar equation

$$\frac{\partial\theta}{\partial t} + u\cdot\nabla\theta + \kappa(-\triangle)^{\alpha}\theta = f,$$
$$\theta|_{t=0} = \theta_0$$

Here $\alpha \in (0,1]$, $\kappa > 0$, $\theta(t)$ is a real function of two space variables $x \in R^2$ and a time variable t. The function $\theta(t) = \theta(x,t)$ represents the potential temperature. The fluid velocity u is determined from θ by a stream function ψ $(u_1, u_2) = \left(-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1}\right)$ where the function ψ satisfies $(-\triangle)^{\frac{1}{2}}\psi = -\theta$

Rates of decay will be obtained in several Sobolev spaces as well as for moments of the solutions. Lower rates will also be discussed. The last question that will be addressed is nonuniform decay for solutions with data in L^2 spaces.

Regularity for suitable weak solutions to the Navier–Stokes equations in critical Morrey spaces

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A class of sufficient conditions of local regularity for suitable weak solutions to the nonstationary three-dimensional Navier–Stokes equations are discussed. The corresponding results are formulated in terms of functionals which are invariant with respect to the Navier–Stokes equations scaling. The famous Caffarelli–Kohn–Nirenberg condition is contained in that class as a particular case.

About solvability and simulation equations describing motion of ideal liquid with free boundary

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Let an inviscid incompressible fluid occupy a domain in the plane (x,y) bounded by the free surface $-\infty < y \leq \eta(x,t)$, $-\infty < x < \infty$, $\quad t > 0$. Assuming that the fluid flow is potential, we have $v(x,y,t) = \nabla\Phi(x,y,t)$.

We have follow equations

$$\Delta\Phi(x,y,t) = 0,$$

$$(\eta_t + \Phi_x\eta_x - \Phi_y)|_{y=\eta(x,t)} = 0,$$

$$\left(\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + gy\right)\Big|_{y=\eta(x,t)} = 0,$$

$$\Phi_y|_{y=-\infty} = 0.$$

We consider equivalent equations, called the Dyachenko's equations, describing nonstationary motion of ideal liquid with free boundary in a gravitational field. Dyachenko's equations are nonlinear integro-differential equations. They turn out to be convenient for numerical modeling.

Existence of analytic solutions of the above equations for a sufficiently small time interval is proved. Solutions from Sobolev spaces of finite order are also investigated.

In the second part of the work, a numerical method for obtaining approximate solutions is constructed. The convergence is proved, provided that a smooth solution exists. An efficient numerical scheme is proposed.

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Analysis of optimal control problems for the two-dimensional thermistor system

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An optimal control problem for the thermistor system is considered. This system is essentially coincides with the coupled system modelling a Poiseuille-type flow of a heat-convergent viscous fluid. First, the precise mathematical problem is established and the proof of existence of the optimal solution is given with appropriate function spaces. Then, Gâteaux differentiability is shown for the thermistor system with respect to control and the optimality system is obtained.

The talk is based on the paper [1].

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Qualitative properties of the 3D Navier–Stokes dynamics

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Let us consider a homogeneous incompressible fluid subject to an external force. Its motion is described by the Navier–Stokes equations

$$\dot{u} + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = f(t,x), \quad \operatorname{div} u = 0, \quad x \in \mathbb{T}^3. \tag{1}$$

Here $u = (u_1, u_2, u_3)$ and p are unknown velocity and pressure fields, f is an external force, $\nu > 0$ is the viscosity, and $\mathbb{T}^3$ is a three-dimensional torus. Equations (1) are supplemented with the initial condition

$$u(0,x) = u_0(x). \tag{2}$$

It is well known that, under suitable regularity assumptions on f and u_0, problem (1), (2) has a weak solution defined for all $t \geq 0$ and a unique strong solution defined on a small time interval $[0,T]$. We wish to study qualitative properties of solutions in the case when the dynamics is governed by a random external force of the form

$$f(t,x) = h(x) + \eta(t,x),$$

where $h \in L^2(\mathbb{T}^3, \mathbb{R}^3)$ is a given function and $\eta(t,x)$ is a stochastic process white in time and smooth in the space variables. Namely, we assume that

$$\eta(t,x) = \sum_{j=1}^{\infty} b_j \dot{\beta}_j(t) e_j(x),$$

where $\{e_j\}$ is a trigonometric basis in $L^2(\mathbb{T}^3, \mathbb{R}^3)$, $\{\beta_j\}$ is a family of independent standard Brownian motions, and $b_j \geq 0$ are some constants satisfying the condition $\sum_j b_j^2 < \infty$. Under these assumptions, it can be shown that Eq. (1) has a weak solution u defined for $t \geq 0$ such that the distribution of $u(t)$ does not depend on t. In this case, the distribution of $u(t)$ is called a *stationary measure* for (1). We prove the following result.

Theorem. *Under the above conditions, there is an integer* $N \geq 1$ *not depending on* h, η, *and* ν *such that if*

$$b_j \neq 0 \quad \textit{for } j = 1, \dots, N,$$

then the following assertions hold for any stationary measure μ*:*

(i) *The support of* μ *coincides with the phase space.*

(ii) *Any finite-dimensional projection of* μ *is minorised by a measure that possesses a positive density with respect to the Lebesgue measure.*

Furthermore, assertions (i) *and* (ii) *remain valid for any weak solution of problem* (1), (2).

The proof of this theorem is based on controllability properties of 3D Navier–Stokes equations established in [4, 5] (see also [1, 2] for the 2D case) and a general result on the image of probability measures under smooth mappings (cf. [3, 6]).

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Spectral portraits of the Orr–Sommerfeld operator with large Reynolds numbers

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First, we investigate a model spectral problem of the form

$$-\varepsilon y'' + iq(x)y = \lambda y, \qquad x \in (a,b) \subset \mathbb{R},$$

where ε and λ are physical and spectral parameters, respectively, and $q(x)$ is the real analytic function. In this model problem $q(x)$ stands for the velocity profile of the Orr–Sommerfeld operator and ε is the reciprocal of the Reynolds number. Adding some boundary conditions we investigate the spectrum behavior of this problem as $\varepsilon \to 0$. We show that under some additional assumptions on $q(x)$ the spectrum asymptotically lies on some analytic curves in the complex plane which we call limit spectral graph. We investigate the nature of these curves and write the formulas for the density of the eigenvalues on the critical curves.

Then, we clarify the connection of the above model problem with the spectral problem for the Orr–Sommerfeld equation. We give a rigorous definition of the Orr–Sommerfeld operator and show that its spectral portraits for large Reynolds numbers are similar to those for the model problem.

The motion of an inextensible thread and incompressible fluid and foundations of mechanics

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1. As it was noted by V. I. Yudovich, the motion equations of an ideal inextensible thread are very similar to the Lagrangian equations of an ideal incompressible fluid. They have the form

$$x_{tt} + (\sigma x_s)_s = 0, \tag{1}$$

$$(x_s, x_s) \equiv 1, \tag{2}$$

where $x(s,t) \in R^n$ is the time-dependent configuration of the thread, $0 \leq s \leq 1,\ t \geq 0$, and $\sigma(s,t)$ is the tension of the thread. If we compare it with the Lagrange equations for the fluid:

$$\frac{\partial^2 g_t(a)}{\partial t^2} + \nabla p(g_t(a), t) = 0, \tag{3}$$

$$\left|\frac{\partial g}{\partial a}\right| \equiv 1, \tag{4}$$

we see that the roles of the tension σ and the pressure p are similar, and the structures of the corresponding systems are similar. Boundary conditions (for free ends) are

$$\sigma(0) = \sigma(1) = 0. \tag{5}$$

This is similar to the case of completely free surface of the fluid (i.e. a fluid drop in the absence of gravity), where the boundary condition is

$$p\Big|_{\partial M} = 0. \tag{6}$$

V. I. Yudovich has proved the following remarkable fact: if $x(s,t)$ is a smooth solution of (1), (2) with the boundary conditions (5), then the tension σ is everywhere positive. However, there is no theorem stating the existence of solution of equations (1), (2) for a long time. Moreover, the computer simulations of the author (presented in the talk) show that the smooth solution fails to exist for a long time. So, we have to introduce generalized solutions.

The thread is *inextensible*, but this does not mean that it is *incompressible*. To the contrary, we can imagine that it can take a form of infinitesimal curls or zigzags, so that its macroscopic linear density is more than 1. Thus, the configuration space of the thread (with free ends, and fixed center of masses) is a *convex* compact $\mathcal{M} \subset X = L^2([0,1],\mathbb{R}^n)$. So, we have the following general problem: define and construct the motion $x(t)$ on the convex set $\mathcal{M}$ in a Hilbert space X satisfying the initial condition $x(0) = x_0,\ \dot{x}(0) = v_0$.

The definition of the permissible motion $x(t)$ coincides with its construction and can be regarded as the use of the d'Alembert's principle. Choose a small $\Delta t > 0$ and define a sequence of points $x_1, x_2, \dots$ in $\mathcal{M}$ which constitute the approximate trajectory. First define $y_1 = x_0 + v_0 \Delta t$. Then take $x_1 \in \mathcal{M}$ the closest point to y_1. If $x_0, x_1, \dots, x_n$ are already found, define $y_{n+1} = x_n + (x_n - x_{n-1})$, and $x_{n+1} \in \mathcal{M}$ the closest point to y_{n+1}. This process can be continued indefinitely. Instead of looking for the limit of the approximate trajectory as $\Delta t \to 0$, we use the strength of the Nonstandard Analysis. Let ${}^*\mathbb{R},\ {}^*X,\ {}^*\mathcal{M}, \dots$ be the nonstandard extensions of $\mathbb{R}, X, \mathcal{M}, \dots$. Take $\Delta t \in^* \mathbb{R}$ *infinitesimally small* and define the above sequence $x_i \in {}^*\mathcal{M}$ for all *nonstandard* integer n. Define the trajectory ${}^*x({}^*t)$ for all nonstandard ${}^*t \geq 0$ (by, say, linear interpolation), and then define the standard trajectory $x(t)$ as a *standard part* $\mathrm{st}({}^*x({}^*t))$ of the approximate nonstandard trajectory ${}^*x({}^*t)$. Our trajectory satisfies the variational inequality

$$\int (\ddot{x}(t), y(t) - x(t))\, dt \geq 0 \tag{7}$$

for any function $y(t) \in \mathcal{M}$. However, this particular solution can lose the kinetic energy.

2. Consider now the motion of an ideal incompressible fluid in a bounded domain $M \subset \mathbb{R}^n$. Classically, the configuration space of the fluid is the group $\mathcal{D}(\mathcal{M}) = \mathcal{D}$ of volume preserving diffeomorphisms of the domain M; it can be regarded as a subset of the Hilbert space $X = L^2(M, \mathbb{R}^n)$. X is the space of a free motion, and $\mathcal{D}$ is the constraint (the system is forced to move along $\mathcal{D}$). $\mathcal{D}$ is not closed, and we have to extend the configuration space. Let Ω be a set endowed with a σ-algebra $\mathcal{F}$ and a probability measure μ. *Generalized configuration* f is a map $\Omega \ni \omega \mapsto x(\omega) \in M$ such that the image of the measure μ is the Lebesgue measure in M. Denote by $\mathcal{G}(\mathcal{M}) = \mathcal{G}$ the space of all generalized configurations; it is naturally a subset of the Hilbert space $Z = L^2(\Omega, \mathbb{R}^n)$. Again, $\mathcal{G}$ is a constraint, and we have to define, what is the motion along the set $\mathcal{G}$ (generalized flow). This motion should satisfy the additional condition: there exists an

incompressible vector field $u(x,t) \in L^2$ such that $\dot{x}(\omega,t) = u(x(\omega,t),t)$ for a. a. (ω,t); in other words, our generalized flow should be *generalized flow with definite velocity (GFDV)*. The problem is similar to the case of the thread, but the set $\mathcal{G}$ is not convex any more, and the above method does not work. The condition GFDV defines a distribution of subspaces $Z_x \subset T_x Z$ along the set $\mathcal{G}$. Using the idea similar to the case of the thread, we construct the trajectory $x(t) \in \mathcal{G}$, $\sqcup \geq \prime$ satisfying the initial condition $\dot{x}(\omega,0) = u_0(x(\omega,0))$ which is a GFDV. Let $u(x,t)$ be the corresponding velocity field.

Theorem. *The vector field $u(x,t)$ is a weak solution of the Euler equations satisfying initial conditions $u(x,0) = u_0(x)$ and such that its kinetic energy $E(t) = (u,u)/2$ is a decreasing function of time.*

Geometric optics for the incompressible Euler equations

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Geometric optics for the linearized Euler equation is a standard tool used to prove existence of shortwave instabilities in stationary ideal fluid flows. In this talk we describe geometric optics method for the full nonlinear incompressible Euler equations in the open space and apply it to prove nonlinear instability in the energy norm for certain classes of equilibria including those with hyperbolic points.

Heat and mass transfer by natural convection from vertical surface in porous media

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This paper deals with the phenomenon of natural convective heat and mass transfer near a vertical surface embedded in a fluid saturated porous medium. An integral method is applied for the solution of the problem where the wall temperature and concentration are the constants. The governing parameters of the flow are the buoyancy ratio N and Lewis number Le. The results for the Local Nusselt number and Local Sherwood number have been compared graphically with Lie and Kulacki* for a wide range of these parameters. An excellent agreement has been found with one another.

The governing equations of the flow field are:

$$f'' = \theta' + N\varphi' \tag{1}$$

$$\theta'' = \frac{1}{3}f'\theta - \frac{2}{3}\theta' \tag{2}$$

$$\varphi'' = Le\left(\frac{1}{3}f'\varphi - \frac{2}{3}f\varphi'\right) \tag{3}$$

with the boundary conditions

$$f(0) = 0,\ \theta(0) = \varphi(0) = 1 \tag{4}$$

$$f'(\infty) = \theta(\infty) = \varphi(\infty) = 0 \tag{5}$$

where f is dimensionless stream function; N is buoyancy ratio; Le is Lewis number; θ is dimensionless temperature; φ is dimensionless concentration and primes denote differentiation with respect to the similarity variable η.

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Weighted Schauder estimates for evolution Stokes problem

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We consider the non-stationary evolution Stokes problem

$$\frac{\partial \mathbf{v}(x,t)}{\partial t} - \nabla^2 \mathbf{v}(x,t) + \nabla p(x,t) = \mathbf{f}(x,t),$$
$$\nabla \cdot \mathbf{v}(x,t) = 0, \quad x \in \Omega, \quad t \in (0,T),$$
$$\mathbf{v}(x,0) = \mathbf{v}_0(x), \qquad \mathbf{v}(x,t)|_{x\in S} = 0,$$

in a bounded or exterior domain $\Omega \subset R^n$, $n \geq 2$, with a smooth boundary S, and we prove the solvability of this problem in weighted anisotropic Hölder spaces $C_s^{l,l/2}(Q_T)$, $Q_T = \Omega \times (0,T)$, with the norm

$$|u|_{C_s^{l,l/2}(Q_T)} = \sup_{0<t<T} t^{(l-s)/2} [u]^{(l,l/2)}_{Q'_t}$$
$$+ \sum_{s<2k+|j|<l} \sup_{0<t<T} t^{(2k+|j|-s)/2} \sup_{\Omega} |D_t^k D_x^j u(x,t)| + |u|_{C^{s,s/2}(Q_T)}$$

where $s \leq l$, $Q'_t = \Omega \times (t/2,t)$, $C^{s,s/2}(Q_T)$, $s \geq 0$ is a standard anisotropic Hölder space, $[u]^{(l,l/2)}_{Q'_t}$ is the principal part of the norm in $C^{l,l/2}(Q't)$. The space $C_s^{l,l/2}(Q_T)$ can be defined also for $s < 0$, in which case the last term in the norm should be omitted. These spaces are well known in the theory of parabolic initial-boundary value problems. Under appropriate assumptions on the data $\mathbf{f}$, $\mathbf{v}_0$, we obtain the solution with the following differentiability properties: $\mathbf{v} \in C_s^{2+\alpha,1+\alpha/2}(Q_T)$, $\nabla p \in C_{s-2}^{\alpha,\alpha/2}(Q_T)$, $\alpha \in (0,1)$. In particular, if $\mathbf{f} = 0$, $s \in [0,2)$, then $\mathbf{v}_0$ should belong to $C^s(\Omega)$ and satisfy only the natural compatibility conditions $\nabla \cdot \mathbf{v}_0 = 0$, $\mathbf{v}_0|_{x\in S} = 0$. This result makes it possible to prove local in time solvability of nonlinear problem under the same assumptions on $\mathbf{v}_0$. It extends to generalized Stokes equations of the form

$$\frac{\partial \mathbf{v}(x,t)}{\partial t} + \mathcal{A}\left(x,t,\frac{\partial}{\partial x}\right) \mathbf{v}(x,t) + \nabla p(x,t) = \mathbf{f}(x,t),$$

where $\mathcal{A}$ is second order strongly elliptic operator.

Singularities of Bernoulli free boundaries

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A Bernoulli free-boundary problem [4] is one of finding domains in the plane on which a harmonic function simultaneously satisfies linear homogeneous Dirichlet and inhomogeneous Neumann boundary conditions. One of the earliest, and still one of the most important, of such problems comes from Bernoulli's theorem and the constant-pressure condition in the study of Stokes waves in hydrodynamics.

In this talk based on the paper [5] we show that, for a large class of Bernoulli problems, a free boundary which is symmetric with respect to a vertical line through an isolated singular point must necessarily have a corner at that point, and we give a formula for the contained angle. The assumptions used admit the possibility of other singular points, even uncountably many, on the free boundary. This result is a substantial extension of that in the theory of hydrodynamic waves [1, 3] on the existence of a corner of 120° at the crest of symmetric Stokes waves of extreme form. We also show that, even in the presence of singularities, any geometrically simple Bernoulli free boundary is necessarily symmetric, extending the result for the water-wave problem in [2].

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Non-autonomous 2D Navier–Stokes system with singularly oscillating external force and its global attractor

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We study the global attractor $\mathcal{A}^\varepsilon$ of the non-autonomous 2D Navier–Stokes (N.–S.) system with singularly oscillating external force of the form $g_0(x,t) + \dfrac{1}{\varepsilon^\rho} g_1\left(\dfrac{x}{\varepsilon}, t\right)$, $x \in \Omega \subset \mathbb{R}^2, t \in \mathbb{R}$, $0 < \rho \leq 1$.

Excluding the pressure we consider the 2D N.–S. system

$$\partial_t u + \nu L u + B(u,u) = P g_0(\cdot, t) + \varepsilon^{-\rho} P g_1(\cdot/\varepsilon, t), \ \operatorname{div} u = 0, \ u|_{\partial\Omega} = 0, \quad (1)$$

where P is the known Leray orthogonal projector from $L_2(\Omega)^2$ onto the space H of divergence free vector field with finite L_2-norm. To begin with, we assume that $g_0(x,t)$, $x \in \Omega$, $t \in \mathbb{R}$, and $g_1(z,t)$, $z \in \Omega$, $t \in \mathbb{R}$ are translation bounded in the spaces $L_2^b(\mathbb{R}; H)$ and $L_2^b(\mathbb{R}; Z)$, respectively. Here $Z = L_2^b(\mathbb{R}_z^2; \mathbb{R}^2)$. The equation (1) generates a process $\{U_\varepsilon(t,\tau),\ t \geq \tau, \tau \in \mathbb{R}\}$ acting in H by the formula $U_\varepsilon(t,\tau)u_\tau = u(t)$, where $u_\tau(\cdot) \in H$, $u(t) = u(\cdot, t), t \geq \tau$, is a solution of equation (1) with initial data $u(\cdot,\tau) = u_\tau(\cdot)$. We show that the process $\{U_\varepsilon(t,\tau)\}$ has the uniform (w.r.t. $\tau \in \mathbb{R}$) global attractor $\mathcal{A}^\varepsilon$ that is bounded in H for every fixed $\varepsilon > 0$. Moreover,

$$\|\mathcal{A}^\varepsilon\|_H \leq C_0 + C_1 \varepsilon^{-\rho}, \ \forall \varepsilon > 0, \ \rho \geq 0, \quad (2)$$

and the constants C_0 and C_1 are independent of ε and ρ. Note that the size of the attractor $\mathcal{A}^\varepsilon$ in the space H may grow up to infinity as $\varepsilon \to 0+$.

Let the function $g_1(z,t)$ have a *divergence representation*

$$g_1(z,t) = \partial_{z_1} G_1(z,t) + \partial_{z_2} G_2(z,t), \ z = (z_1, z_2) \in \mathbb{R}^2, \ t \in \mathbb{R}, \quad (3)$$

where the functions $G_j(z,t) \in L_2^b(\mathbb{R}; Z)$ for $j = 1, 2$. Then we prove the theorem on the uniform boundedness of global attractors $\mathcal{A}^\varepsilon$ with respect to $\varepsilon \in]0,1]$:

$$\|\mathcal{A}^\varepsilon\|_H \leq C_2, \ \forall \varepsilon \in]0,1]. \quad (4)$$

We consider the "limiting" N.–S. system

$$\partial_t u + \nu L u + B(u,u) = P g_0(\cdot, t), \ \operatorname{div} u = 0, \ u|_{\partial\Omega} = 0. \quad (5)$$

We study the deviation $w(x,t) = u^\varepsilon(x,t) - u^0(x,t), t \geq \tau$, of a solution $u^\varepsilon(x,t)$ of equation (1) from a solution $u^0(x,t)$ of equation (5) with the same initial data $u^\varepsilon(x,\tau) = u^0(x,\tau)$. If the function $g_1(z,t)$ satisfies the above divergence condition, then we prove the following estimate:

$$\|w(t)\|_H \leq \varepsilon^{(1-\rho)} C e^{r(t-\tau)},\ \forall \varepsilon, 0 < \varepsilon \leq 1,$$

where the constants C and r are independent of ε and $0 \leq \rho \leq 1$.

If the functions $g_0(x,t)$ and $g_1(z,t)$ are *translation compact* in the corresponding spaces and the function $g_1(z,t)$ has a divergence representation (3), then we prove the global attractors $\mathcal{A}^\varepsilon$ of equation (1) converges to the global attractor $\mathcal{A}^0$ of the "limiting" equation (5) in H as $\varepsilon \to 0+$, that is

$$\mathrm{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \to 0\ (\varepsilon \to 0+),$$

where $\mathrm{dist}_H(X,Y)$ denotes the Hausdorff distance from a set X to a set Y in the space H.

Let the Grashof number $G := \lambda_1^{-1}\nu^{-2}\left\|g^0\right\|_{L_2^b(\mathbb{R};H)}$ of the "limiting" N.–S. system (5) is sufficiently small. (Here λ_1 is the first eigenvalue of the Stokes operator L.) Then the global attractor $\mathcal{A}^0$ is exponential and

$$\mathrm{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \leq C(\rho)\varepsilon^{1-\rho},\ \forall \varepsilon \in]0,1]. \tag{6}$$

The results of my talk is based on our joint work with V. V. Chepyzhov.

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Relativistic generalization of quasi-Chaplygin equations

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The nonlinear evolution of many unstable media is represented in long-wave approximation by quasi-Chaplygin equations (QCEs)

$$d\vec{v}/dt = c_0^2 m \nabla \rho_*^{1/m},\ d\rho_*/dt + \rho_* div\vec{v} = 0,\ d/dt = \partial/\partial t + (\vec{v}\nabla), \qquad (1)$$

where ρ_* is an effective density, v is a velocity, c_0 is the “sound” speed, m is the parameter called an azimuth number. These equations differ from the ideal gas equations only in that they contain negative compressibility. Such instabilities are encountered in nature quite often; for instance, about 50 such examples are considered in [1], where a general theory of the QCEs is formulated. The properties of the system of QCEs were studied systematically latter in [2]. A distinguishing feature of this nonlinear equations is that they have analitic solutions in 1D case and 2D (stationary) case.

In this paper, we propose a relativistic generalization of QCEs (1) and demonstrate its relationship with a number of physical problem (Chaplygin gas, Van der Waals gas in an unstable domain, a cylinder of liquid with the surface tension, and a plasma pinch). These equations have the form:

$$\gamma d(\gamma\vec{v})/dt = -\nabla p_{ef} - (\gamma/c)^2 \vec{v} dp_{ef}/dt,\ \partial(\rho_* u^k)/\partial x^k = 0, \qquad (2)$$

where $p_{ef} = -c_0^2 m \rho_*^{1/m}$, x^k, and u^k are the 4-vectors of coordinates and velocity, $u^k = (\gamma, \gamma\vec{v}/c)$, c is the velocity of light, $\gamma = (1-v^2/c^2)^{-1/2}$. In this generalization we proceeded from the equations of one-liquid relativistic fluid dynamics [3] (we disregard pressure in the formula for enthalpy). An analytic solution to these nonlinear equations is obtained for 1D case.

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Attractors for motion equations of viscoelastic media

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We shall describe sufficient conditions for existence of minimal trajectory attractors and global attractors of autonomous evolution equations in Banach spaces without assumptions of any invariance of the trajectory space of an equation. We shall also describe sufficient conditions for existence of minimal uniform trajectory attractors and uniform global attractors of non-autonomous equations. Here it is not assumed that the symbol space of an equation is a compact metric space and that the family of trajectory spaces corresponding to this symbol space is translation-coordinated or closed in any sense. Then we shall discuss some properties of the attractors and apply the results to the motion equations of an incompressible viscoelastic medium with the Jeffreys constitutive law:

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i} + grad\ p = Div\ \sigma + f, \tag{1}$$

$$\sigma + \lambda_1 \left(\frac{\partial \sigma}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial \sigma}{\partial x_i} \right) = 2\eta \left(\mathcal{E} + \lambda_2 \left(\frac{\partial \mathcal{E}}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial \mathcal{E}}{\partial x_i} \right) \right), \tag{2}$$

$$div\ u = 0, \tag{3}$$

$$u\Big|_{\partial\Omega} = 0. \tag{4}$$

Here Ω is an arbitrary bounded domain in R^n $(n = 2, 3)$, u is an unknown vector of velocity of points of the medium, p is an unknown function of pressure, σ is an unknown deviator of the stress tensor, f is the given body force (all of them depend on a point $x \in \Omega$ and a moment of time t), $\mathcal{E} = \mathcal{E}(u) = (\mathcal{E}_{ij})$, $\mathcal{E}_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right)$ is the strain velocity tensor, $\eta > 0$ is the viscosity of the medium, λ_1 is the relaxation time, λ_2 is the retardation time, $0 < \lambda_2 < \lambda_1$. The gradient *grad* and the divergence *div* are taken with respect to the variable x. The divergence *Div* of a tensor is the vector with the coordinates $(Div\ \sigma)_j = \sum\limits_{i=1}^{n} \dfrac{\partial \sigma_{ij}}{\partial x_i}$. The density of the medium is considered to be equal to one. We construct minimal trajectory, global, minimal uniform trajectory and uniform global attractors for weak solutions of this boundary value problem.

This talk is based on joint works with V. Zvyagin. The research was partially supported by grants 04-01-00081 (Russian Foundation of Basic Research), VZ-010-0 (Ministry of Education and Science of Russia and CRDF) and MK-3650.2005.1 (Grant of President of Russian Federation).

Existence and linear stability of solitary waves in coupled KdV equations

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We consider a system of coupled equations of KdV type which is a model for interactions of long waves, for example in a stratified fluid. We are interested in the existence and linear stability of solitary waves for the system. Previous studies in this direction have relied primarily on the fact that solitary waves for this system are minimizers of an energy functional under a certain constraint. We prove the existence of a variety of solitary waves several of which are not constrained minimizers using perturbative means. We also compute the spectra of the linearizations about these solutions. We discover that the phenomenon known as "leapfrogging" (that is, a solution which looks like a solitary wave in each component which oscillate about a shared center of mass) is linked to an oscillatory instability in the problem.

Schwartz derivative and its applications

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It is well known, that there are two types of bifurcations of the fixed point of diffeomorfisms and flows loss of stability, namely soft bifurcations and hard ones. To identify these bifurcations type, it is necessary to take into account the sign of the corresponding normal form cubic member. In 1985, Sataev suggested a multidimensional analogue of well known Schwartz derivative (Schwartzian) , allowing to distinguish between the types of bifurcations in question [1].

This result is a part of the author's extensive research. The main purpose of this work is to describe the methodology of obtaining multidimensional Schwartzian. This work can be useful for any specialist dealing with dynamic systems and bifurcation theories. In the course of the research many applications of multidimensional Schwartzian were considered. The main conclusion drawn from the research is that in the case of bulky dynamic systems calculation the multidimensional Schwartz derivative is easier than using normal forms. Furthermore, the undertaken research has led to the following hypothesis: Hypotheses (Sataev, Yakushkin) There exists a parameter manifold, such that for any parameter values from that parameter manifold Henon (Henon-like) mapping has only soft period-doubling bifurcations.

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Free-surface hydrodynamics in conformal variables

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Euler equations for potential flow of 2D ideal fluid with free surface in presence of gravitational field are studied. The equations could be written in a compact form after conform mapping of the domain, covered with fluid to a horizontal strip (if the depth is finite), or on the half-plane (if it is a case of infinite depth).

Evolution equations have non-canonical Hamiltonian structure and are very convenient for numerical simulation. We studied in details only the case of infinite depth. The equations are written in terms of functions, which are analytic in the lower half-plane. They have singularities in the upper half-plane; the temporal dynamics of these singularities is a question of critical importance. We show that for symmetric standing waves one can construct such a solution that singularities are posed on the periodical set of cuts, parallel to the imaginary axis. Jumps of the conformal shape of surface and complex velocity on these cuts obey the system of non-local equations, which resemble equations of 1D gas dynamics. They could be integrated if the cut is narrow. Moreover, the Euler equation in conformal variables has a set of “hidden” motion constants, associated with zeros of function $R = \partial w/\partial z$ (z and w are physical and conformal variables). This could mean that the whole system is integrable, however this fact is not proven yet. The numerical simulation demonstrates the behaviour that is typical for integrable systems.

Spatialy non-decaying solutions of Navier–Stokes equations in cylindrical domains

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The existence and dissipativity of weak solutions for two and three dimensional Navier–Stokes equations in cylindrical domains in the class of spatially non-decaying functions are verified. In particular, this class of solutions contains all Poiseuille flows and all known structures bifurcating from them. In addition, in two-dimensional case, the uniqueness and further regularity of such solutions are also proven.

We also study the long-time behavior of these solutions. In two-dimensional case, we prove the existence of smooth locally compact global attractor which usually has an infinite Hausdorff and fractal dimensions and possesses the typical (for dissipative systems in unbounded domains) upper and lower bounds for the Kolmogorov's epsilon entropy.

Finally, in the 3D case, due to the lack of uniqueness, we use the so-called trajectory dynamical systems approach and construct the associated global attractor for the shift semigroup in the appropriate phase space of trajectories of the 3D Navier–Stokes equations considered.

The global in time behavior of the symmetric compressible Navier–Stokes–Poisson flows with a free boundary

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We consider symmetric flows of a viscous compressible barotropic fluid or gas around a hard core with a free boundary, for arbitrarily large initial data. The flows are driven by a general mass force f_S (depending both on the Eulerian and Lagrangian mass coordinates) and an outer pressure $p_{\Gamma,S}$. The case of the self-gravitation arising in astrophysics [1] is covered. For a general non-monotone state function p, we prove the uniform-in-time energy bound, the uniform bounds for the density ρ and the free radius R and the stabilization of the kinetic and potential energies as $t \to \infty$. We also obtain H^1-stabilization of the velocity v to zero provided that the second viscosity is zero. For non-decreasing p, we study L^q-stabilization of ρ and the stabilization of R together with the corresponding ω-limit set in the general case of non-unique stationary solution possibly with zones of vacuum. In the case of increasing p and of stationary densities ρ_S separated from zero, we establish the uniform-in-time H^1-bounds and the uniform stabilization for ρ and v.

Furthermore, for a general increasing p, we investigate the existence and the uniqueness of stationary solutions as well as the static stability of these with positive ρ_S, and we present a variational study of stationary solutions and of their static stability in terms of the potential energy. In the astrophysical context, we prove that, in particular, the stationary solution is unique and statically stable provided that the first adiabatic exponent is greater or equals 4/3. In the case where the ω-limit set for ρ and R contains the statically stable positive stationary solution, we derive the uniform stabilization to this one and then establish the stabilization rate bounds for ρ and v in L^2 and H^1 of the exponential type as $t \to \infty$ by constructing new non-trivial Lyapunov functionals for the problem. Moreover, we prove that the statically stable stationary solutions are exponentially asymptotically stable, and this nonlinear dynamic stability is in addition stable with respect to small non-stationary perturbations of f_S and $p_{\Gamma,S}$. We also introduce a variational condition on the stationary solution ensuring the global with respect to the data dynamic stability. The results are proved both in the Eulerian coordinates and in the Lagrangian mass ones. They are contained in [2–5].

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Approximation-topological methods of the study in problems of hydrodynamics of non-Newtonean media

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Approximate-topological ideas form the base of the proposed method. Its short scheme is the following. At first we give the operator treatment of the considered problem in the form of the operator equation in some functional space which is appropriate for this problem. For this equation we suggest a family of auxiliary operator equations defined in a functional space which has (as a rule) better topological properties. Further, the application of some versions of the topological degree theory in Banach spaces and a priori estimates allow us to obtain the solvability of these auxiliary equations. At last, the passage to the limit (in the sense of distributions theory) in the sequence of solutions of auxiliary equations, based on a priori estimates of these solutions in the initial functional space, leads to the solution of the principal operator equation and, hence, to the solution of the initial problem. The description of this method, given as the example of the investigation of the solvability of the initial-boundary value problem for Navier–Stokes system is presented in [1].

The application of this method allows to solving new boundary value and initial boundary value problems for stationary and evolution problems of non-Newton hydrodynamics (see [2–4]). On the base of this method, we study some optimal feedback control problems (see [5, 6]). This method is useful also in the study of attractors of weak solutions to equations of non-Newton hydrodynamics (see [7]).

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Разрывы со стационарной, периодической и стохастической структурой в нелинейной среде с дисперсией

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Методы ранее разработанной теории бездиссипативных разрывов обобщаются на слабодиссипативный случай. Анализируются численные решения задачи об эволюции начальных данных типа сглаженной ступеньки (задача о распаде произвольного начального разрыва) для обобщенного уравнения Кортевега—де Вриза со старшей производной пятого порядка и дополнительным диссипативным членом, коэффициент при котором мал:

$$a_t + b_1 a_x + (a^2/2)_x + b_3 a_{xxx} + b_5 a_{xxxxx} = \epsilon a_{xx},$$

$$a|_{t=0} = a_1\{1 - \tanh[(x - \Delta)/l)]\}/2.$$

Посредством варьирования амплитуды разрыва, т. е. величины a_1, обнаружено, что с течением времени устанавливается стационарное, периодическое по времени, или стохастическое решение. В стационарном случае решение содержит волновые зоны, описываемые некоторыми усредненными уравнениями. Эти зоны могут быть одноволновыми, двухволновыми или трехволновыми с рациональным отношением периодов взаимодействующих волн. Переходы между волновыми зонами или однородными состояниями, интерпретируются как бездиссипативные разрывы внутри слабодиссипативной структуры. Имеется много различных типов таких разрывов, но все их можно отнести к двум классам: бездиссипативные разрывы со структурой (аналоги ударной волны в газовой динамике) и бифуркации (аналоги слабых разрывов). Система решений повторяет строение ранее исследованного так называемого резонансного дерева (множества ограниченных стационарных решений уравнения Кортевега—де Вриза с производной пятого порядка). Это связано с тем, что стационарные резонансные решения играют роль притягивающих решений в нестационарных процессах. В одних случаях в результате такого притяжения слабодиссипативное решение со временем становится стационарным, при этом локально любой фрагмент этого решения можно рассматривать как стационарное бездиссипативное решение. В этом случае с физической точки зрения решение интерпретируется как решение с каустикой и безотражательным разрывом. В других случаях такое притяжение приводит к возникновению периодических режимов. С физической точки зрения такое решение можно рассматривать как решение с отраженной волной на разрыве. Последующие многократные отражения приводят к возникновению захваченных волн и к общему резонансу в динамической системе. Кроме того, имеются стохастические решения, обусловленные притяжением к большому числу стационарных решений (многократные отражения волн в этом случае приводят к хаосу). Обнаружены гистерезисы: в зависимости от характера эволюции (выбора сглаживания начальных данных, изменения коэффициентов уравнения или граничных условий в процессе установления) конечное установившееся решение может быть различным.

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Диффузия вихревого слоя: автомодельность, задача с неизвестной границей, устойчивость

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Автомодельные решения, зависящие от переменной типа $x/\sqrt{t}$, являются классическими фундаментальными решениями одномерных линейных и нелинейных уравнений теплопроводности и описывают многочисленные физические явления с разрывом на границе в начальный момент времени. Однако автомодельность — свойство не только данного уравнения, но всей начально-краевой задачи, поставленной для него.

В работе для задачи о диффузии вихревого слоя обсуждаются классы сред и типы задания граничных условий, при которых автомодельные решения существуют. Для вязкопластической среды в полуплоскости в классе непрерывно дифференцируемых во всей полуплоскости функций автомодельность отсутствует, и задача сводится к задаче с неизвестной границей. Её постановка следующая. В области $D = \{0 < y < 1; t > 0\}$ необходимо найти две функции: $\sigma(y,t)$ — касательное напряжение в зоне течения и $\xi(t)$ — толщину зоны течения, удовлетворяющие одному уравнению

$$\sigma_{yy} + \xi\dot{\xi}y\sigma_y = \xi^2\sigma_t \tag{1}$$

с граничными и начальными условиями

$$\sigma(0,t) = S_0, \quad \sigma(1,t) = 1, \quad \sigma_y(1,t) = 0, \quad \xi(0) = 0 \tag{2}$$

где S_0 — заданное на границе постоянное касательное напряжение, отнесённое к пределу текучести при сдвиге ($|S_0| > 1$).

Приводятся методы анализа, асимптотики и интегральные оценки решения задачи (1), (2) в различных функциональных пространствах. Даются оценки параметров устойчивости такого течения относительно малых начальных возмущений.

О численной стабилизации неустойчивого течения Куэтта по граничным условиям

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При исследовании динамики вязкой несжимаемой жидкости исследователи нередко сталкиваются с явлением неустойчивости течений. Рассмотрим уравнения Навье—Стокса в области между двумя бесконечными соосными цилиндрами, из которых внутренний вращается с некоторой постоянной угловой скоростью, а внешний покоится. Ограничимся лишь решениями, периодическими вдоль оси цилиндра и не зависящими от угла поворота относительно оси. Хорошо известно одно стационарное решение этой задачи – это течение Куэтта. Известно также (см. [3]), что при фиксированном периоде и достаточно больших значениях числа Рейнольдса существует другое решение стационарной задачи, называемое вихрями Тейлора. Такая картина хорошо согласуется с физическими экспериментами, в которых течение Куэтта с ростом числа Рейнольдса становится неустойчивым и происходит его бифуркация в вихри Тейлора. Такое же поведение неустойчивого течения Куэтта установлено при численном моделировании (см. [2]).

В работе предлагается и реализуется дискретный аналог алгоритма стабилизации по граничным условиям, который для уравнений Навье—Стокса изначально был разработан и теоретически обоснован в дифференциальном случае А. В. Фурсиковым в [1]. Задача стабилизации неустойчивого решения формулируется как задача построения таких краевых условий, при которых возмущение $\mathbf{u}(\mathbf{x}, t)$ будет подавляться с наперед заданной скоростью, и, следовательно, решение будет стремиться к течению Куэтта. Скорость стабилизации, определяется некоторой постоянной $\alpha > 0$:

$$\|\mathbf{u}(\mathbf{x}, t)\|_{\mathbf{L}_2(\Omega)} \leq C \exp(-\alpha t). \tag{1}$$

Сам алгоритм состоит из трех этапов: продолжение – проектирование заданного начального условия из исходной области в расширенную область, интегрирование нестационарной системы уравнений в расширенной области, приводящее к искомым граничным условиям, и, собственно, стабилизация в исходной области с полученными граничными условиями.

Работа является первой попыткой стабилизации неустойчивости, наблюдаемой в естественных условиях и хорошо описываемой математической моделью. Стабилизация с наперед заданной скоростью не представляется пока возможной, однако построение стабилизирующих краевых условий, дающее некоторую достаточно высокую скорость, можно считать успешным. Впервые в численной стабилизации применяется алгоритм с обратной связью, хотя в дифференциальном случае использование обратной связи является ключевым моментом.

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Разложение гармонического подпространства в $L_p(G)$ в сумму аналитического и коаналитического подпространств

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Пусть $G \subset \mathbb{R}^2$ — ограниченная связная область с гладкой границей Г. Пространство $\mathbb{R}^2$ будем стандартно отождествлять с комплексной плоскостью $\mathbb{C}$. Определим функциональные пространства аналитических и антианалитических функций соответственно как

$$\mathcal{O}_p(G) = \{u \in L_p(G): \ \partial_{\bar{z}} u = 0 \ \text{ в } \ \mathcal{D}'(G)\}\,,$$
$$\overline{\mathcal{O}}_p(G) = \{u \in L_p(G): \ \partial_z u = 0 \ \text{ в } \ \mathcal{D}'(G)\}\,.$$

Здесь $\partial_{\bar{z}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial x}\right)$ — оператор Коши-Римана, а оператор $\partial_z = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial x}\right)$ сопряжен с ним. Пусть $1 < p < \infty$.

Утверждается, что пространство $\mathcal{O}_p(G) + \overline{\mathcal{O}}_p(G)$ — замкнутое подпространство в $L_p(G)$. Этот факт эквивалентен справедливости неравенства

$$\|u\|_{L_p(G)/\mathbb{C}} \leq M\, \|\nabla u\|_{\left(W_p^{-1}(G)\right)^2}\,,$$

где $M > 0$ - константа, которое при $p = 2$ известно как неравенство LBB (Ладыженской-Бабушки-Брецци). Доказательство справедливости LBB неравенства при $1 < p < \infty$ проводится методами, рассматриваемыми в [2]. Замкнутость пространства $\mathcal{O}_p(G) + \overline{\mathcal{O}}_p(G)$ означает, что операция восстановления аналитической функции по ее вещественной или мнимой части, известная из классической теории функций комплексного переменного (см. [3]), устойчива в норме $L_p(G)$. Отметим, что вещественные и мнимые части функций из $\mathcal{O}_p(G)$ принадлежат пространству $\mathcal{O}_p(G) + \overline{\mathcal{O}}_p(G)$.

В случае односвязной области G, имеет место равенство $H_p(G) = \mathcal{O}_p(G) + \overline{\mathcal{O}}_p(G)$, где $H_p(G)$ — подпространство гармонических функций в $L_p(G)$. Декомпозицию можно продолжить на все $L_p(G)$ следующим образом:

$$L_p(G) = \big(\mathcal{O}_p(G) + \overline{\mathcal{O}}_p(G)\big) \oplus \Delta \overset{\circ}{W}{}_p^2(G).$$

Последнее разложение при $p = 2$ известно (см. [1]).

В случае многосвязной области, пространство $\mathcal{O}_p(G) + \overline{\mathcal{O}}_p(G)$ является замкнутым подпространством в $H_p(G)$. Например, для области в виде кольца $r < |z| < R$ при $p = 2$ имеет место разложение

$$H_2(G) = \big(\mathcal{O}_p(G) + \overline{\mathcal{O}}_p(G)\big) \oplus L\{e_1\}$$

где $e_1(z) = \ln|z| - \iint_G \ln|z|\,dx\,dy,\ L\{e_1\} = \{\alpha e_1,\ \alpha \in \mathbb{C}\}$.

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Условия существования сильного решения нелинейных эволюционных уравнений типа уравнений Навье—Стокса

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В сепарабельном гильбертовом пространстве H рассмотрим задачу

$$u' + Au + B(u,u) = f(t), \quad u(0) = 0, \quad t > 0 \tag{1}$$

где $B(u,g)$ — билинейный оператор, $f(t)$ — функция со значениями в H, а A - самосопряженный оператор с вполне непрерывным обратным. Через $H_{p,\theta}(0,a)$ обозначим пространство функций f со значениями в H, таких, что $|A^\theta f(\eta)|^p$ суммируема на $(0,a)$. Здесь $|\cdot|$ норма в H. Будем говорить, что задача (1) сильно разрешима, если при любом $a > 0$ из $f \in H_{p,\theta}(0,a)$ вытекает $u' + Au \in H_{p,\theta}(0,a)$. Известная проблема о гладкости решения уравнений Навье—Стокса сводится к вопросу сильной разрешимости задачи (1).

Потребуем выполнения следующего условия

$$|A^{-\gamma}B(u,g)| \leq c\left[|A^{\gamma_0}u||A^{\gamma_0+\frac{1}{2}}g| + |A^{\gamma_0}g||A^{\gamma_0+\frac{1}{2}}u|\right], \tag{2}$$

при всех γ_0 и γ удовлетворяющих условиям $2\gamma_0 = 2\delta - \gamma$, $-\infty < 4\gamma \leq 3$, $\delta > 0$, где число δ постоянное, а число c зависит от γ.

Определение. Функция $f(t) \in H_{p,\theta}(0,a)$, $|f|_{H_{p,\theta}(0,a)} \neq 0$, называется разделяющей функцией задачи (1), если выполнены (a) и (b):

(a) решение $u(t)$ задачи (1) в $(0,a)$ существует и

$$u' + Au \in H_{p,\theta}(0, a - \varepsilon)$$

при любом $\varepsilon \in (0,a)$, но $u' + Au \notin H_{p,\theta}(0,a)$;

(b) если $g(t) \in H_{p,\theta}(0,a)$ и $|g|_{H_{p,\theta}(0,a)} < |f|_{H_{p,\theta}(0,a)}$, то решение задачи

$$v'(t) + Av + B(v,v) = g(t), \quad v(0) = 0, \quad 0 < t < a$$

существует и $v'(t) + Av \in H_{p,\theta}(0,a)$.

Теорема. *Предположим, что выполнены условие* (2) *и неравенства*

$$0 \le \delta - \frac{\theta}{2} < \frac{1}{2}, \ \left(\delta - \frac{\theta}{2}\right) p' < 1, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Тогда для того, чтобы задача (1) *не была сильно разрешимой в* $H_{p,\theta}$ *необходимо и достаточно, чтобы при некотором* $a > 0$ *существовала в* $H_{p,\theta}(0, a)$ *разделяющая функция задачи* (1).

Проекция Чепмена—Энскога и проблемы Навье—Стокс приближения кинетических уравнений

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Целью доклада является исследование проблемы Навье—Стокс приближения кинетических уравнений в терминах проекции Чепмена—Энскога. Одна из основных проблем моментной теории описания процессов неравновесной термодинамики связана со специфическими трудностями для систем моментов — часть неизвестных задачи (неравновесные переменные — моменты высших порядков) не имеют интуитивного физического смысла. Такие переменные не могут быть определены из эксперимента [1, 4]. Тогда что значат для них данные Коши и тем более граничные данные?

Предложенный Чепменом и Энскогом подход [2] позволяет остаться в рамках начальных и граничных условий только для консервативных переменных, поскольку суть подхода состоит в нахождении операторной зависимости неравновесных переменных от базовых, консервативных величин, т.е. нахождении проекции из фазового пространства моментных аппроксимаций в фазовое пространство консервативных переменных.

В докладе, для моментных аппроксимаций кинетического уравнения Больцмана-Пайерлса [3] будет доказано существование проекции Чепмена-Энскога [5] в фазовое пространство консервативной переменной (диффузионная мода [6]) и в фазовое пространство физических переменных (проекция второй скорости звука).

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Численные методы решения некоторых краевых задач аэродинамики с обобщенными граничными условиями

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Необходимость постановки краевых задач с обобщенными граничными условиями возникла в аэродинамике при моделировании обтекания тел при наличии отсоса потока, осуществляемого на их поверхности через точечное отверстие или узкую щель. При разработке математической модели, учитывающей отсос потока, сначала была рассмотрена задача о потенциальном стационарном обтекании тонких и телесных объектов в рамках модели идеальной несжимаемой жидкости. В ряде работ (см. [1]) такие задачи рассматривалась с теоретической и прикладной точки зрения на основе подхода, при котором поле скоростей должно иметь в окрестности точки отсоса (линии отсоса) особенность с заданной асимптотикой.

В предлагаемом докладе излагается новый подход к постановке и решению задач указанного типа. Основная идея этого подхода заключается в том, чтобы рассматривать краевые значения неизвестных функций и их нормальных производных как обобщенные функции. Так в задаче о потенциальном обтекании тела с отсосом потока в некоторой точке, краевое значение нормальной составляющей вектора скорости задается равным дельта-функции Дирака с носителем в этой точке.

Для того чтобы придать задаче строгую математическую постановку, вводится понятие обобщенных краевых значений и обобщенных нормальных производных, которые рассматриваются как обобщенные функции, заданные на границе области. При этом обобщенные функции понимаются как линейные функционалы над некоторым пространством основных функций. Исследованы вопросы разрешимости возникающих краевых задач и построены численные схемы их решения типа метода дискретных вихрей. Для плоских задач и для трехмерной задачи на плоском экране (задача о потенциальном обтекании плоской несущей поверхности) доказана сходимость получаемых численных решений к точным в некотором специальном смысле [2].

На основе указанной теории сформулирована задача о потенциальном обтекании тонких и телесных объектов в виде краевой задачи обобщенным граничным условием, предложен и обоснован численный метод решения задачи [3]. Важной особенностью численного алгоритма является то, что задача сводится к решению системы линейных уравнений, причем матрица этой системы в точности та же самая, что и в задаче без отсоса потока, а требуемая особенность в решении автоматически возникает за счет специального вида правой части этой системы.

Далее осуществляется эмпирическое соединение разработанного метода решения стационарной задачи и известного метода вихревых рамок численного решения нестационарной задачи об обтекании крыла идеальной несжимаемой жидкостью с развитием вихревого следа. На основе разработанной математической модели было проведено численное моделирование обтекания прямоугольного крыла при наличии отсоса потока через тонкую щель или точечное отверстие. Исследовалась структура вихревого следа при различных положениях устройства, осуществляющего отсос потока. При численном моделировании обнаружено, что при определенных положениях устройств, осуществляющих отсос потока, происходит потеря устойчивости концевых вихрей и начинается процесс их разрушения на расстояниях от крыла в несколько раз меньших, чем без отсоса.

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Моделирование поведения потоков крови в сосудах с патологиями

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В настоящее время остро стоит проблема определения влияния патологии сосудов на характеристики кровотока. В данной работе исследования проводились для двух типов патологий: наличие атеросклеротической бляшки и извитость кровеносных сосудов. Моделирование проводилось на основе решения уравнений Навье—Стокса в трехмерной постановке. Стенки сосудов принимались жесткими, что оправдано для атеросклеротически пораженных сосудов. Данная система решалась численно методом конечных элементов. Алгоритм решения — UMFPACK.

Численное исследование течения крови в артериях с тромбозом показало, что наблюдается значительное увеличение скорости и, как следствие, образование области с пониженным давлением, что приводит к турбулизации потока и возможному образованию кавитационных каверн. Следует отметить, что турбулизация потока как непосредственно способствует разрушению бляшки, так и играет роль катализирующего фактора при возникновении кавитационных явлений. Увеличение касательных напряжений при турбулизации потока приводит к разрушению макромолекул, в обычном случае стабилизирующих поток и способствует возникновению дополнительной неустойчивости. Риск образования кавитационных каверн также возрастает при гипертермии потока в пораженном сосуде, что имеет большое прикладное значение (например, это следует учитывать при повышении температуры после операции). Полученные распределения гидродинамических характеристик потока позволяют оценить механизмы деструктивного воздействия на бляшку:

1) в области с пониженным давлением может возникать кавитация, паровые пузырьки переносятся в область с большим давлением вниз по потоку (на задней стенке), схлопываться, что приводит к изъязвлению бляшки;

2) возникновение турбулентного пятна на границе вихревой области приводит к тому, что касательные напряжения могут увеличиваться на два порядка.

Исследовано поведение потоков крови в шунтированных кровеносных сосудах. Рассчитаны оптимальные геометрические характеристики сосудов-протезов, которые шунтируют стенозированный сосуд: угол, под которым он вшивается; радиус кривизны; соотношение диаметров шунтируемого и шунтирующего сосудов и т. п. Выбор оптимальных характеристик шунтирующего сосуда позволяет существенно уменьшить риск послеоперационных осложнений.

Исследовано поведение потоков крови в извитых сосудах при различных радиусах кривизны. Показано, что смыкание сосудов приводит к увеличению областей отрывных течений, и, как следствие, потере потоком устойчивости. Дальнейшее развитие вихревых структур приводит к резкому снижению расхода, а значительная деформация давления приводит к возникновению обратных течений, что блокирует нормальный кровоток в пораженном сосуде.

Проведено моделирование течения крови в сосуде после удаления атеросклеротической бляшки. Результаты расчетов показывают, что резкое расширение проходного сечения приводит к значительной неоднородности распределения по области течения. Наблюдается значительное падения давления после прохождения конфузорного участка. Данный эффект будет проявляться сильнее в том случае, когда амплитуда движения незакрепленных участков будет выше.

Полученные результаты являются существенным продвижением в исследовании задач моделирования поведения потоков крови в сосудах человека при различных патрологиях и являются теоретической основой для новых методов диагностики и показаний к оперативному вмешательству при сосудистых заболеваниях.

Динамически равновесные формы мезомасштабных вихрей в океане

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Мезомасштабные вихри интрузионного происхождения в океане наблюдаются в виде вращающихся водных масс, отличающихся от окружающей фоновой воды по плотности, солёности и температуре. Можно отметить два типа таких вихрей. Первый тип — это затопленные на глубине $\sim$1 км антициклонически (против часовой стрелки) вращающиеся линзы. Однородное по плотности ядро таких линз имеет горизонтальные размеры $\sim$20 км и высоту $\sim$500 м. Орбитальные горизонтальные скорости в них имеют порядок 30 см/с. Такие линзы, содержащие средиземноморскую воду, наблюдаются в северной Атлантике. Они медленно двигаются преимущественно на юго-запад и живут до 7—10 лет.

Второй тип — это образующиеся в результате меандрирования фронтальных океанских течений антициклонические и циклонические приповерхностные ринги, например, ринги Гольфстрима и Куросио. Горизонтальные размеры рингов достигают сотен километров, вертикальный — сотен метров, а орбитальные скорости в них до 1 м/с. Время их жизни от нескольких месяцев до 2 лет.

По мнению авторов, долгая жизнь мезомасштабных вихрей связана с близостью реальных вихрей к их динамически равновесным формам.

Рассматриваются задачи о нахождении динамически равновесных форм однородной по плотности вращающейся массы идеальной жидкости (линзы) и приповерхностного ринга, погруженных в покоящийся стратифицированный океан на вращающейся Земле. Эти вихревые образования представляют собой своеобразные жидкие природные гироскопы, прецессирующие в инерциальном пространстве из-за вращения Земли. Учитываются и вертикальная и горизонтальная проекции скорости вращения Земли.

Из условия равенства давлений на поверхности раздела получены уравнения для форм поверхности раздела водных масс. Точное аналитическое решение задачи для линзы и антициклонического ринга в линейно стратифицированном океане в окрестности их залегания показывает, что при реальных параметрах явления динамически равновесная форма поверхности раздела представляет собой наклоненный к горизонту трехосный эллипсоид, близкий к наклоненному эллипсоиду вращения. Поверхность раздела циклонического ринга — часть наклоненного к горизонту двуполостного эллиптического гиперболоида или эллиптического конуса.

Свободные поверхности рингов — части эллиптических параболоидов: антициклонического — возвышающегося над невозмущенной поверхностью океана («горб»), а циклонического — расположенного ниже поверхности океана (углубление), что соответствует наблюдениям поверхностей рингов со спутников.

О механизме потери устойчивости вязкого течения в прямолинейной трубе

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Проведены исследования возможности реализации стационарных и нестационарных стабилизированных прямолинейных течений вязкой баротропной среды. Показано, что в этом случае при наличии градиента давления система уравнений Навье—Стокса не допускает решений, описывающих стабилизированное, прямолинейное движение. Полученный результат дает основание заключить, что при течении реальных сред ламинарные потоки противоречат закону сохранения массы и могут реализоваться только как метастабильные состояния движения. При движении реальных сред с изменением давления вдоль линии тока плотность не остается постоянной. В ламинарном потоке такое изменение плотности приводит к рассогласованию в выполнении закона сохранения массы, а само ламинарное течение из-за этого приобретает неустойчивый метастабильный характер, в котором объемная упругая деформация выступает постоянно действующим возмущающим фактором. Возможность существования такого метастабильного состояния движения объясняется тем, что в отличие от несжимаемой жидкости в сжимаемых средах существуют степени свободы, связанные с объемной упругой деформацией для возбуждения которых необходимо затратить некоторое дополнительное количество энергии на сжатие среды. При ламинарном течении степени свободы, связанные с объемной упругой деформацией находятся в невозбужденном, «замороженном» состоянии, а сама среда проявляет свойства несжимаемой жидкости. В результате ламинарное течение оказывается энергетически более выгодным по сравнению с абсолютно устойчивым состоянием движения сжимаемой среды. При этом критерий Рейнольдса определяет не только устойчивость ламинарного потока к внешним возмущениям, но выступает и в качестве меры вероятности возбуждения акустических возмущений в потоке вязкой жидкости.

В линейном приближении рассмотрена задача о стабилизированном течении вязкой баротропной жидкости в прямой трубе. Показано, что в этом случае задача не имеет стационарных решений. При этом нестационарное возмущение параметров течения, прежде всего, связано с возникновением упругой волны давления, распространяющихся с постоянной скоростью по направлению течения жидкости.

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Нестационарная нелинейная модель взаимодействия гравитационных поверхностных волн с внутренней волной

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Нелинейная стационарная модель была рассмотрена в [1–3] при этом оставался открытым вопрос устойчивости решения. Линейная нестационарная модель была изучена в [4]. В настоящей работе исследуется нелинейная нестационарная модель взаимодействия гравитационных поверхностных волн, обладающих групповой скоростью c_g, с внутренней волной, фазовая скорость которой равна c. Решается одномерная задача Коши для гравитационных поверхностных волн с волновым числом k и частотой ω на неоднородном поверхностном течении

$U(x)$ в бесконечно глубокой жидкости [5, 6].

$$k_t + \omega_x = 0,$$
$$\left(\frac{a^2}{k^{1/2}}\right)_t + \left[\frac{a^2}{k^{1/2}}\left(\frac{g^{1/2}}{2k^{1/2}} + v(x)\right)\right]_x = 0,$$
$$k(x,0) = k_0 = \text{const},$$
$$a(x,0) = a_0 = \text{const},$$
$$\omega = g^{1/2}k^{1/2}\left(1 + \frac{1}{2}\,a^2k^2\right) + kv(x),$$
$$v(x) = -c + \varepsilon U(x),$$

где a — амплитуда волн, c — фазовая скорость внутренней волны в системе координат, связанной с ней, $U(x)$ — скорость подводного течения. Экспериментально показано, что скорость течения мала по сравнению со скоростью внутренней волны, это позволяет ввести малый параметр $\varepsilon = \max[U(x)/c]$. Используя метод возмущений, получены выражения первого порядка, описывающие эволюцию приведенной частоты $\Omega = \sqrt{k}$ и волнового действия $a^2/\sqrt{k}$. Из полученных решений следует, что в результате взаимодействия характеристики поверхностной волны изменяются: волновое действие растет и волна становится короче если $c > 3c_g/2$, где c — фазовая скорость внутренней волны, а c_g — групповая скорость поверхностных волн. При $c_g < c < 3c_g/2$ волновое действие уменьшается и волна укорачивается. Если $c < c_g$, то волновое действие уменьшается и волна удлиняется. В случае относительно крутых поверхностных волн (квадрат их крутизны $(ak)^2 > 2/9$) в результате взаимодействия возникают три возмущения на поверхности, которые сходны по форме с внутренней волной. Одно из них стационарно (т. е. неподвижно в системе координат, связанной с внутренней волной). Поведение двух других возмущений зависит от соотношения между c_g и c. В случае относительно пологих поверхностных волн ($(ak)^2 < 2/9$) возникают всего лишь две волны возмущения. Одна из них стационарна, другая может быть предвестником (при $c_g > c$) или следом внутренней волны (при $c_g < c$).

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Влияние структурных особенностей на движение двух гидродинамически взаимодействующих частиц в поле градиента температуры

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Большой научный и практический интерес вызывает установление закономерностей в поведении аэрозольных частиц в неоднородных по температуре газах. Если размеры частиц велики по сравнению со средней длиной свободного пробега молекул газа, то газ рассматривается как сплошная среда и применяются методы гидродинамики. Движение частицы, взвешенной в газе, находящегося в поле градиента температуры (термофоретическое движение), описано во многих работах. Актуальным является обобщение существующих моделей с учетом важных эффектов. Известный интерес представляет теория движения двух гидродинамически взаимодействующих частиц с учетом поверхностных явлений и объемных особенностей частиц.

Нами реализована идея объединения эффектов гидродинамического взаимодействия двух одинаковых крупных твердых сферических частиц и неоднородности их по теплопроводности для описания термофоретического движения вдоль линии центров частиц с учетом скачка температуры на их поверхности. Поля температур, скоростей и давления ищутся в виде рядов, поиск неопределенных коэффициентов этих рядов сводится к поиску векторов в банаховом пространстве. Решения записываются через ограниченные линейные операторы, отображающие это пространство в себя. Получены формулы для силы Стокса, термофоретической силы и скорости, для поправок, характеризующих влияние вышеперечисленных эффектов. Так как точные формулы записаны через бесконечномерные операторы, то для практических расчетов получены приближенные формулы, записанные через определенным образом усеченные операторы. С помощью этих формул составлены таблицы поправок к скорости термофореза.

Contents (Содержание)